The Observability Radius of Network Systems

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Topics in this talk

1. Observability radius: from classical to networks
   - Motivation: adversary perturbations
   - The observability radius of linear systems
   - The observability radius of network systems

2. An algorithm for the observability radius

3. The role of topology: networks with random weights

4. Attacks on power systems
Network of dynamical systems $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
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Perturbations against observability

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- monitored by sensor nodes $\mathcal{O} \subseteq \mathcal{V}$,
- chosen so that the dynamics is observable

An adversary can manipulate some edges $\mathcal{M} \subseteq \mathcal{E}$

Can the adversary make the dynamics unobservable?

How large the perturbation must be?
- this defines a notion of robustness
Perturbations and observability of linear systems

Before perturbation, \((A, C)\) is observable

\[
x(t + 1) = Ax(t) \quad y(t) = Cx(t)
\]

The observability radius is

\[
\mu(A, C) = \min_{\Delta A, \Delta C} \left\| \begin{bmatrix} \Delta A \\ \Delta C \end{bmatrix} \right\|_2,
\]

s.t. \((A + \Delta A, C + \Delta C)\) is unobservable

Typical result (real perturbations): \(\mu(A, C) = \min_{s \in \mathbb{R}} \sigma_n \left( \begin{bmatrix} sI - A \\ C \end{bmatrix} \right)\)


**Shortcomings:**

- unstructured: \(\Delta_A\) and \(\Delta_C\) are full matrices
- 2-norm does not quantify the effort of an attacker
- both \(A\) and \(C\) are perturbed
Our problem: perturbations of dynamical networks

Localized observation matrix:
\[ O = \{ o_1, \ldots, o_p \} \quad \text{and} \quad C_O = \begin{bmatrix} e_{o_1} & \cdots & e_{o_p} \end{bmatrix}^T \]

The network observability radius is

\[
\min_{\Delta} \| \Delta \|_F^2,
\]

s.t. \((A + \Delta, C_O)\) is unobservable

\[
\Delta \cdot M = 0
\]

where
- structure is imposed: \( \cdot \) is entrywise product, \( M_{ij} = 0 \) if \((i, j) \in M\)
- Frobenius norm \( \| \Delta \|_F^2 = \sum_{i,j} \delta_{ij}^2 \) is chosen
- only \( A \) is perturbed
Computing the observability radius
Computing the observability radius

More explicitly:

\[
\begin{align*}
\min_{\Delta, \lambda, x} \quad & ||\Delta||^2_F \\
\text{s.t.} \quad & C_0 x = 0 \\
& (A + \Delta) x = \lambda x \\
& ||x||_2 = 1 \\
& \Delta \cdot M = 0
\end{align*}
\]

Frobenius norm
unobservability
eigenvalue constraint
normalization
structural constraint

- Not convex
- Feasible if $M = \mathcal{E}$
- Since $(A, C)$ is observable, $\Delta$ must be nonzero
Idea for an algorithm

Two phases:

1st phase: Fix $\lambda$ and solve

$$\min_{x,\Delta} \|\Delta\|_F^2$$

s.t. $C_Ox = 0$

$$(A + \Delta)x = \lambda x$$

$$\|x\|_2 = 1$$

$$\Delta \cdot M = 0$$

2nd phase: Search for the best $\lambda \in \mathbb{C}$

Exhaustive search seems unavoidable:

Derivation of the algorithm

1. Incorporate structural constraints in $\|\Delta\|^2_F$ (approximately)

\[
\text{cost} \longrightarrow \sum_{i=1}^{n} \sum_{j=1}^{n} \delta_{ij}^2 (1 - m_{ij})^{-1}
\]

2. Decompose $\lambda = \lambda_R + i\lambda_I$ and divide real and imaginary parts

3. Define Lagrange multipliers for the other constraints and write $\nabla \mathcal{L} = 0$

4. Rewrite as generalized non linear eigenvalue problem:

Finding $\sigma, z$ such that $Hz = \sigma K_z z$

5. Solve iteratively by “freezing” the nonlinearity $K_z$ (inverse iteration method)

If convergent, it gives a suboptimal solution

Based on ideas from

Networks with random weights
The effect of disconnecting cuts

Define the minimal observability-preventing perturbation as

$$\delta := \min_{\lambda, x, \Delta} \|\Delta\|_F$$

s.t. $$C_{\mathcal{O}} x = 0$$

$$(A + \Delta)x = \lambda x$$

$$\|x\|_2 = 1$$

$$\Delta \cdot M = 0 \quad (M = \mathcal{E})$$

If $a_{ij}$ are independent random variables uniformly distributed in $[0, 1]$, then

$$\mathbb{E}[\delta] \leq \frac{\Gamma(1/k)}{\sqrt{k}} \frac{\Gamma(\omega + 1)}{\Gamma(\omega + 1 + 1/k)}$$

where $\Gamma(\bullet)$ is the Gamma function

- $\Omega_k(\mathcal{O})$ is a collection of disjoint cuts of size $k$, where each cut disconnects a non-empty subset of nodes from $\mathcal{O}$
- $\omega = |\Omega_k(\mathcal{O})|$ is the number of such cuts
Example of application

Looking for disconnecting cuts...
Example of application

Looking for disconnecting cuts... 

Note: 
- The number of disconnecting cuts is essential:

\[
\frac{\Gamma(1/k)}{\sqrt{k}} \frac{\Gamma(\omega + 1)}{\Gamma(\omega + 1 + 1/k)} \geq 0.8 \frac{1}{\omega + 1}, \quad \text{increasing in } k
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Example of application

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- To have a small bound, we need many small cuts
Example of application

Looking for disconnecting cuts...

\[ k = 2, \omega = 4 \implies \mathbb{E}[\delta] \leq \frac{\Gamma(1/2)}{\sqrt{2}} \frac{\Gamma(5)}{\Gamma(5 + 1/2)} = 0.5747 \]

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- To have a small bound, we need many small cuts
- Often, the best choice is just isolating single nodes
Large networks

Consider a sequence of networks with increasing size $n \to \infty$:
If $\omega \to \infty$ and $k$ constant, then

$$\frac{\Gamma(1/k) \Gamma(\omega + 1)}{\sqrt{k} \Gamma(\omega + 1 + 1/k)} \sim \frac{\Gamma(1/k)}{\sqrt{k}} \frac{1}{(\omega + 1)^{1/k}}$$

The network becomes less robust to perturbations as the size of the network increases, with a rate determined by $k$
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The network becomes less robust to perturbations as the size of the network increases, with a rate determined by $k$

Questions:
- Is the bound tight?
- How do optimal perturbations look like?
The role of graph topology: Examples

Line network

\[ \text{Line is strongly structurally observable} \]
\[ \Rightarrow \text{Best perturbation } \delta \text{ is disconnecting} \]
\[ \delta = \max \{a_{i,i+1}\} \]

\[ \mathbb{E}[\delta(n)] = \frac{1}{n} \]

the bound is tight
The role of graph topology: Examples

Line network

\[ \delta = \max_i \{a_{i,i+1}\} \]

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The bound is tight

Star network

Best perturbation

\[ \min_{i\geq 2, i\neq j} \frac{|a_{ii} - a_{jj}|}{\sqrt{2}} \]

introduces an artificial symmetry

\[ E[\delta(n)] \sim \frac{1}{\sqrt{2} n^2} \quad \text{as } n \to \infty \]
The role of graph topology: Examples

**Line network**

Line is strongly structurally observable

⇒ Best perturbation $\delta$ is disconnecting

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**Star network**

Best perturbation $\min_{i,j \geq 2, i \neq j} \frac{|a_{ii} - a_{jj}|}{\sqrt{2}}$

introduces an artificial symmetry

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Real example
Attacks on power systems

IEEE 14 power grid observed from bus 1

Small-signal model is linear descriptor system

\[
\begin{bmatrix}
I & 0 & 0 \\
0 & M_g & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\delta} \\
\dot{\omega} \\
\dot{\theta}
\end{bmatrix}
= -
\begin{bmatrix}
0 & -I & 0 \\
S_{gg} & D_g & S_{gl} \\
S_{lg} & 0 & S_{ll}
\end{bmatrix}
\begin{bmatrix}
\delta \\
\omega \\
\theta
\end{bmatrix}
+ \begin{bmatrix}
P_\omega \\
P_\theta
\end{bmatrix}
\]

\(\delta\): generator rotor angles
\(\omega\): generator rotor frequencies
\(\theta\): voltage angles at the buses

Goal: inducing an unobservable unstable mode

<table>
<thead>
<tr>
<th>Perturbation</th>
<th>(|\Delta|_F)</th>
<th>Unobservable mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disconnect load 1 (([S_{ll}]_{1,2} = 0))</td>
<td>4.60</td>
<td>10.92</td>
</tr>
<tr>
<td>Stop generator 1 ((\dot{\delta}_1 = 0))</td>
<td>2.59</td>
<td>10.92 ± 20.95j</td>
</tr>
<tr>
<td>Modify impedance (53 lines modified)</td>
<td>2.34</td>
<td>10.92 ± 10^4j</td>
</tr>
</tbody>
</table>

Creating artificial dynamical symmetries seems to require smaller perturbations than disconnecting the network
Conclusion

Summary

1. New robustness measure of network systems
2. Heuristic algorithm for its solution
3. Different types of graphs $\rightarrow$ different observability radii

Note: everything can be translated to controllability
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Summary

1. New robustness measure of network systems
2. Heuristic algorithm for its solution
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Note: everything can be translated to controllability

Open problems

1. Effective computation of $\delta$
2. Lower bound on $\mathbb{E}[\delta]$ (beyond connectivity)
3. Role of topology: more examples, different random models
4. Explore real examples