Available online at www.sciencedirect.com
ScienceDirect
Systems & Control Letters 57 (2008) 474–481
systems & control letters
www.elsevier.com/locate/sysconle

A new vertex result for robustness problems with interval matrix uncertainty

T. Alamo[a], R. Tempo[b,*], D.R. Ramírez[a], E.F. Camacho[a]

[a] *Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla, Escuela Superior de Ingenieros, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain*
[b] *IEIIT-CNR, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy*

Received 3 November 2006; received in revised form 13 November 2007; accepted 17 November 2007
Available online 29 January 2008

Abstract

This paper[1] addresses a family of robustness problems in which the system under consideration is affected by interval matrix uncertainty. The main contribution of the paper is a new vertex result that drastically reduces the number of extreme realizations required to check robust feasibility. This vertex result allows one to solve, in a deterministic way and without introducing conservatism, the corresponding robustness problem for small and medium size problems. For example, consider quadratic stability of an autonomous n_x dimensional system. In this case, instead of checking $2^{n_x^2}$ vertices, we show that it suffices to check 2^{2n_x} specially constructed systems. This solution is still exponential, but this is not surprising because the problem is NP-hard. Finally, vertex extensions to multiaffine interval families and some sufficient conditions (in LMI form) for robust feasibility are presented. Some illustrative examples are also given.

Keywords: Interval matrices; Vertex results; Robustness problems; Linear matrix inequalities

1. Introduction

The analysis and synthesis problems in the presence of interval uncertainty have received the attention of the robust control community for decades. Different approaches and results can be found depending on how the uncertainty is modelled. For example, the celebrated Kharitonov's theorem [9] is one of the results that can be used to study stability when interval uncertainty affects the coefficients of a given characteristic polynomial (see [1] for details).

A different approach is required if the interval uncertainty affects the $n_x \times n_x$ matrix A that defines the state space dynamics of a linear system $\dot{x} = Ax$. More specifically, consider the problem of checking robust nonsingularity of A when each component of A lies in a given interval. In [15] it is shown that it suffices to check a subset of the vertex set consisting of 4^n extreme matrices. Robust nonsingularity is closely related to robust stability, see e.g. [1] for details, by means of Kronecker operations, but, Kronecker operations do not preserve the interval matrix structure. Therefore, for robust stability, unfortunately, vertex results hold only for very special classes of systems which include the case of symmetric matrices. In this particular case, in [7] it is shown that the number of extreme matrices required to check robust stability is 2^{n_x-1}. See also [16] for a related result involving an interval Sylvester equation.

A more general class of analysis problems can be addressed using the μ framework [20]. Robustness analysis of a system subject to interval matrix uncertainty can be often formulated as an equivalent μ problem. As stated in [19], purely real μ problems, involving real matrices and only scalar uncertainties, can be solved exactly checking the vertices of the interval matrix. In [18] it is shown that the so-called full structured μ problem involving an $n \times n$ interval real matrix can be solved in an exact way checking 4^n vertices. That is, the results presented in [18] allow one to reduce the number of required vertices from 2^{n^2} to 2^{2n}.

Consider now the problem of checking the robust satisfaction of a linear matrix inequality involving interval

* Corresponding author. Tel.: +39 011 564 5408; fax: +39 011 564 5429.
E-mail addresses: alamo@cartuja.us.es (T. Alamo), roberto.tempo@polito.it (R. Tempo), danirr@cartuja.us.es (D.R. Ramírez), eduardo@cartuja.us.es (E.F. Camacho).

[1] A preliminary version of this paper has been published in the Proceedings of the European Control Conference, Kos, Greece, July 2007.

0167-6911/$ - see front matter
doi:10.1016/j.sysconle.2007.11.003

matrices. Since this problem has been shown to be NP-hard (see [12] and [13]), one cannot expect to find a polynomial-time exact solution. If the interval uncertainty enters in an affine way in the linear matrix inequality, then it suffices to check the vertices of the interval matrix. As the number of vertices of an $n \times n$ matrix is 2^{n^2}, checking all the vertices is possible only when n is very small.

An established line of research deals with the computation of the gain matrix K such that the closed loop system $\dot{x} = (A + BK)x$ is robustly stable and admits a common quadratic Lyapunov function for all the possible realizations of the uncertainty. It is well known, however, that the "common quadratic Lyapunov function" approach is conservative and parameterized Lyapunov functions may reduce this conservatism, see [1]. If the matrices A and B (of dimensions $n_x \times n_x$ and $n_x \times n_u$ respectively) are affected by unstructured interval uncertainty, the synthesis problem can be formulated as a convex problem with $2^{n_x(n_x+n_u)}$ constraints (one for each extreme realization of the uncertainty) see, e.g. [4,8]. The main drawback of this approach is that the number of constraints is not manageable even for relatively small problems. For example, if $n_x = 5$ and $n_u = 2$ then the number of constraints is $2^{35} > 10^{10}$.

When the number of constraints is too large to obtain an exact deterministic solution, different strategies can be adopted. For example, using scaling variables it is possible to bound the effect of the interval uncertainty. In this case, an approximate conservative solution to the problem is obtained [2,3]. Another possibility is to resort to randomized algorithms [14,17], which are proved to converge with probability one to a feasible solution (in case the robust synthesis problem is feasible). See [5] and [11] for examples of randomized algorithms to compute a solution to uncertain linear matrix inequalities.

The main contribution of this paper is a new vertex result that drastically reduces the number of extreme realizations required to solve the synthesis problem under interval uncertainty. For example, with the results of this paper only $2^{2n_x+n_u} = 2^{12} = 4096$ systems are required to solve the aforementioned synthesis problem. This means that when the dimension of the system is relatively small, an exact deterministic solution can be obtained in a reasonable computational time. A generalization of this result is also provided showing in particular that an extreme point result holds for multiaffine interval families. Some sufficient (conservative) results to further reduce the computational time required to check if a given matrix inequality is satisfied for the family of interval matrices are also given in this paper.

The paper is organized as follows: Section 2 presents some notation. The problem statement is introduced in Section 3. The generality of the proposed robust feasibility problem is illustrated by means of motivating examples in Section 4. The new vertex result is presented in Section 5. This result is generalized to multiaffine families in Section 6. Some sufficient conditions for robust feasibility are given in Section 7. Three numerical examples are given in Section 8. The paper draws to a close in Section 9.

2. Notation

- Given vector x, $x(i)$ denotes its i-th component.
- Given a matrix H, $H(i, j)$ denotes its ij-th component.
- $\mathbb{R}^{n\times m}$ denotes the space of $n \times m$ real matrices.
- Given matrix $R \in \mathbb{R}^{n\times m}$, with all its entries nonnegative, and matrix $\tilde{X}$, $\mathrm{I}_c(R, \tilde{X})$ denotes the following interval matrix centered at $\tilde{X}$: $\mathrm{I}_c(R, \tilde{X}) = \{X \ : \ |X(i, j) - \tilde{X}(i, j)| \leq R(i, j), \ \forall i, j\}$. In this context, R is usually referred to as "perturbation scale matrix".
- Given matrix $R \in \mathbb{R}^{n\times m}$, with all its entries non negative, $\mathrm{I}_c(R)$ denotes the following interval matrix (centered at $\tilde{X} = 0$) : $\mathrm{I}(R) = \mathrm{I}_c(R, 0) = \{X \ : \ |X(i, j)| \leq R(i, j), \ \forall i, j\}$.
- The set of $n\times n$ diagonal matrices with diagonal entries equal to 1 or -1 is denoted with $\boldsymbol{\Delta}_n$. That is, $\boldsymbol{\Delta}_n = \{\Delta \in \mathbb{R}^{n\times n} \ : \ \Delta$ is diagonal and $\Delta(i, i) \in \{-1, 1\}, \ i = 1, \ldots, n\}$.
- Given a matrix (vector) H, $|H|$ denotes the matrix (vector) composed by the absolute values of the entries of matrix (vector) H.
- Given a symmetric matrix A, $\lambda_{\max}(A)$ denotes its largest eigenvalue and $\lambda_{\min}(A)$ its smallest eigenvalue; $A > 0$ denotes that A is positive definite and $A < 0$ that it is negative definite; given symmetric matrices A and B, $A > B$ denotes that $A - B$ is positive definite.
- The Euclidean norm is denoted as $\|\cdot\|_2$.

3. Problem statement

As will be shown in the next section by means of some illustrative examples, the robust constraints appearing in the robustness problems with interval uncertainty can often be rewritten in the general form

$$F(\mathrm{X}) + G + G^T + HQ(\mathrm{X}) + Q^T(\mathrm{X})H^T < 0, \quad \forall G \in \mathrm{I}(M), \forall H \in \mathrm{I}(N) \tag{1}$$

where:

(i) $\mathrm{X} = \{X_1, X_2, \ldots, X_t\}$ denotes a subset of the matrix decision variables of the synthesis problem.
(ii) $F(\mathrm{X}) \in \mathbb{R}^{n\times n}$ is a symmetric matrix that is not affected by uncertainty. It is assumed that this matrix is an affine function of the decision variables $\mathrm{X} = \{X_1, X_2, \ldots, X_t\}$.
(iii) $Q(\mathrm{X}) \in \mathbb{R}^{m\times n}$ is an affine matrix function of the decision variables $\mathrm{X} = \{X_1, X_2, \ldots, X_t\}$ that is not affected by uncertainty.
(iv) $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times m}$ are perturbation scale matrices having all elements non negative.

4. Motivating examples

The robustness problems addressed in this paper encompass a large number of robust control synthesis problems. This is illustrated in this section by means of three classical problems: quadratic stabilization of continuous interval systems, L_2 gain minimization in the presence of interval matrix uncertainty and receding horizon control of uncertain discrete-time systems, see e.g. [4,10].

4.1. Quadratic stabilization of continuous-time interval systems

Consider the system $\dot{x} = Ax + Bu$ where $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times n_u}$ are uncertain matrices in the classes $\mathcal{A} = \mathrm{I}_c(R_A, \tilde{A})$ and $\mathcal{B} = \mathrm{I}_c(R_B, \tilde{B})$, respectively. The synthesis problem consists in finding $P = P^T > 0$ and the feedback control law $u = Kx$ such that the closed loop system $\dot{x} = (A + BK)x$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^T Px) < 0, \quad \forall x \neq 0, \forall A \in \mathcal{A}, \forall B \in \mathcal{B}.$$

In particular, a lower bound on the decay rate

$$\lim_{t\to\infty} \mathrm{e}^{\alpha t}\|x(t)\|_2 = 0$$

is guaranteed if

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^T Px) < -2\alpha x^T Px, \quad \forall x \neq 0, \forall A \in \mathcal{A}, \forall B \in \mathcal{B}.$$

This is immediately rewritten as the matrix inequality

$$P(A + BK) + (A + BK)^T P < -2\alpha P, \quad \forall A \in \mathcal{A}, \forall B \in \mathcal{B}.$$

Pre and post-multiplying the previous matrix inequality by P^{-1} and making the change of variable $W = P^{-1}$, $Y = KP^{-1}$ it results that the synthesis problem is rewritten as the following generalized eigenvalue problem

$$\begin{aligned} &\max_{W,Y,\alpha} \ \alpha \\ &s.t. \quad W > 0 \\ &\qquad AW + WA^T + BY + Y^T B^T + 2\alpha W < 0, \\ &\qquad \forall A \in \mathcal{A}, \ \forall B \in \mathcal{B}. \end{aligned}$$

Note that the robust constraint of this problem can be rewritten in the general form given by Eq. (1) by means of the following assignments

$$X_1 = W, \ X_2 = Y, \ t = 2, \ M = 0, \ N = \begin{bmatrix} R_A & R_B \end{bmatrix},$$
$$F(\mathrm{X}) = \tilde{A}X_1 + X_1\tilde{A}^T + \tilde{B}X_2 + X_2^T\tilde{B}^T + 2\alpha X_1,$$
$$Q(\mathrm{X}) = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

4.2. L_2 gain minimization

Consider the following system

$$\begin{cases} \dot{x} = Ax + Bu + Ew \\ y = Cx + Du \end{cases}$$

where $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $E \in \mathbb{R}^{n_x \times n_w}$, $C \in \mathbb{R}^{n_y \times n_x}$ and $D \in \mathbb{R}^{n_y \times n_u}$ are affected by interval uncertainty. That is,

$$\begin{aligned} A \in \mathcal{A} &= I_c(R_A, \tilde{A}) \\ B \in \mathcal{B} &= I_c(R_B, \tilde{B}) \\ E \in \mathcal{E} &= I_c(R_E, \tilde{E}) \\ C \in \mathcal{C} &= I_c(R_C, \tilde{C}) \\ D \in \mathcal{D} &= I_c(R_D, \tilde{D}). \end{aligned}$$

The objective of the synthesis problem is to find a state feedback gain K such that the L_2 gain of the system is minimized. It is well known (see, for example [4]) that the L_2 gain of the closed loop uncertain system is bounded by γ if there exists $P = P^T > 0$ such that for all possible realizations of the uncertain matrices

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^T Px) + y^T y - \gamma^2 w^T w \leq 0.$$

Setting $W = P^{-1}$ and $Y = KW$, the synthesis problem can be immediately rewritten as

$$\begin{aligned} &\min_{W>0,Y,\gamma} \ \gamma^2 \\ &s.t. \begin{bmatrix} AW + WA^T + BY + Y^T B^T & * & * \\ E^T & -\gamma^2\mathrm{I} & * \\ CW + DY & 0 & -\mathrm{I} \end{bmatrix} < 0 \\ &\quad \forall A \in \mathcal{A}, \forall B \in \mathcal{B}, \forall C \in \mathcal{C}, \forall D \in \mathcal{D} \end{aligned}$$

where each entry "$*$" denotes the submatrix required to force the symmetry of the matrix. The robust constraint appearing in this L_2 gain minimization problem can also be rewritten in the general form given in Eq. (1). This is achieved by means of the following assignments

$$X_1 = W, \ X_2 = Y, \ t = 2,$$
$$F(\mathrm{X}) = \begin{bmatrix} \tilde{A}X_1 + X_1\tilde{A}^T + \tilde{B}X_2 + X_2^T\tilde{B}^T & * & * \\ \tilde{E}^T & -\gamma^2\mathrm{I} & * \\ \tilde{C}X_1 + \tilde{D}X_2 & 0 & -\mathrm{I} \end{bmatrix}$$
$$M = \begin{bmatrix} 0 & * & * \\ R_E^T & 0 & * \\ 0 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} R_A & R_B \\ 0 & 0 \\ R_C & R_D \end{bmatrix},$$
$$Q(\mathrm{X}) = \begin{bmatrix} X_1 & 0 & 0 \\ X_2 & 0 & 0 \end{bmatrix}.$$

4.3. Receding horizon control of uncertain discrete-time systems

Consider the time-varying discrete-time system $x_{k+1} = A_k x_k + B_k u_k$ where $A_k \in \mathbb{R}^{n_x \times n_x}$ and $B_k \in \mathbb{R}^{n_x \times n_u}$ are uncertain matrices in the classes $\mathcal{A} = \mathrm{I}_c(R_A, \tilde{A})$ and $\mathcal{B} = \mathrm{I}_c(R_B, \tilde{B})$ respectively for every sample time k. Given an initial condition x_0, semidefinite positive matrices Q and R, and the control policy $u_k = Kx_k$, the value of the cost function

$$J = \sum_{k=0}^{\infty} x_k^T Qx_k + u_k^T Ru_k$$

depends on the particular realization of the uncertain matrices A_k, B_k, $k = 0, \ldots, \infty$. It is well known that if $P = P^T > 0$ and

$$(A_k + B_k K)^T P(A_k + B_k K) - P < -Q - K^T RK, \quad \forall k \geq 0 \qquad (2)$$

then an upper bound of the worst-case value of the cost function is given by $x_0^T Px_0$ (see, for example, [10]). Since $A_k \in \mathcal{A}$ and

$B_k \in \mathcal{B}$ for all $k \geq 0$, the robust constraint (2) is satisfied if

$$(A+BK)^T P(A+BK) - P < -Q - K^T RK, \quad \forall A \in \mathcal{A}, \forall B \in \mathcal{B}. \tag{3}$$

Therefore, the synthesis problem consists in obtaining $P > 0$ and K such that $x_0^T P x_0$ is minimized under the robust constraint given by Eq. (3). Note that it is easy to consider additional constraints, expressed in LMI form, and which are not affected by uncertainty, to deal with bounds on both the state vector x_k and the control input u_k [10].

This control strategy is applied in a receding horizon scheme. That is, at each sample time k, the matrices $P > 0$ and K are obtained such that $x_k^T P x_k$ is minimized under the robust constraint (3). Once this synthesis problem is solved, the control law $u_k = Kx_k$ is applied to the system. Denoting $W = P^{-1}$ and $Y = KW$, the synthesis problem at sample time k can be rewritten as

$$\min_{W>0,Y,\gamma} \gamma$$

$$s.t. \begin{bmatrix} W & WA^T + Y^T B^T & WQ^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ * & W & 0 & 0 \\ * & * & \mathrm{I} & 0 \\ * & * & * & \mathrm{I} \end{bmatrix} > 0,$$

$$\forall A \in \mathcal{A}, \forall B \in \mathcal{B}$$

$$\begin{bmatrix} \gamma & x_k^T \\ x_k & W \end{bmatrix} > 0.$$

The robust constraint appearing in this synthesis problem is rewritten in the general form given by Eq. (1) by means of the following assignments: $X_1 = W$, $X_2 = Y$, $t = 2$,

$$F(\mathrm{X}) = -\begin{bmatrix} X_1 & X_1\tilde{A}^T + X_2^T\tilde{B}^T & X_1 Q^{\frac{1}{2}} & X_2^T R^{\frac{1}{2}} \\ * & X_1 & 0 & 0 \\ * & * & \mathrm{I} & 0 \\ * & * & * & \mathrm{I} \end{bmatrix},$$

$$M = 0, \quad N = \begin{bmatrix} 0 & 0 \\ R_A & R_B \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{4}$$

$$Q(\mathrm{X}) = -\begin{bmatrix} X_1 & 0 & 0 & 0 \\ X_2 & 0 & 0 & 0 \end{bmatrix}.$$

5. Main result: From rectangular interval matrices to binary diagonal matrices

As previously discussed, due to the interval nature of the uncertainty and the affine parametric dependence, it suffices to check all the extreme realizations of the uncertainty. Therefore, in an interval matrix with n rows and m columns in which all the entries of the perturbation scale matrix R are different from zero, 2^{nm} extreme matrices should be considered. However, from the computational point of view, the problem with this vertex approach is that to check feasibility of the robust constraint (1), a huge number of extreme realizations is generally required.

In particular, we observe that in the robust constraint (1), there are $2^{n(n+m)}$ vertex matrices. Hence, for the quadratic stabilization problem this leads to $2^{n_x(n_x+n_u)}$ extreme realizations, whereas for the L_2 gain problem it leads to $2^{n_x(n_x+n_u+n_w)+n_y(n_x+n_u)}$ vertices. Consider, for example, a small dimensional L_2 gain design problem with $n_x = 3$, $n_u = n_w = n_y = 1$. In this case, $2^{19} > 10^5$ extreme realizations should be considered. On the other hand, for a medium size problem with $n_x = 5$, $n_u = n_w = n_y = 2$, the number of required extreme realizations is $2^{59} > 10^{17}$. Clearly, more powerful extreme point results are required in order to solve (in an exact way) the robust synthesis problem for small and medium size problems.

In this section it is shown that it suffices to consider binary diagonal matrices instead of interval rectangular ones. The main advantage of reformulating the robust synthesis problem by means of uncertain diagonal matrices is the following: the number of uncertain parameters required to describe the uncertain matrices grows linearly (instead of quadratically) with the dimension of the uncertain matrices. This implies that the number of extreme realizations of the uncertainty required to check if the robust constraints are satisfied is drastically reduced.

The following theorem constitutes the main contribution of the paper and states that it suffices to consider binary diagonal matrices when addressing the general robust constraint given in Eq. (1). In turn this leads to a major computational improvement as shown in Corollary 1 at the end of this section.

Theorem 1. *Given the matrices F and Q, consider the robust constraint*

$$F + G + G^T + HQ + Q^T H^T < 0, \quad \forall G \in \mathrm{I}(M), \forall H \in \mathrm{I}(N) \tag{5}$$

where $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times m}$. Then, the robust constraint (5) is satisfied if and only if

$$F + \Delta_0(M + M^T)\Delta_0 + \Delta_0 N \Delta_1 Q + Q^T \Delta_1 N^T \Delta_0 < 0, \quad \forall \Delta_0 \in \boldsymbol{\Delta}_n, \forall \Delta_1 \in \boldsymbol{\Delta}_m. \tag{6}$$

Proof. • Robust constraint (5) implies (6):

Suppose that the robust constraint (5) is satisfied. It will be shown that the constraint (6) is satisfied for every $\Delta_0 \in \boldsymbol{\Delta}_n$, $\Delta_1 \in \boldsymbol{\Delta}_m$. Denote $\bar{G} = \Delta_0 M \Delta_0$, $\bar{H} = \Delta_0 N \Delta_1$. From the definition of $\boldsymbol{\Delta}_n$ and $\boldsymbol{\Delta}_m$ it results that

$$|\bar{G}(i,j)| = |\Delta_0(i,i)M(i,j)\Delta_0(j,j)| = M(i,j), \ \forall i,j;$$

$$|\bar{H}(i,j)| = |\Delta_0(i,i)N(i,j)\Delta_1(j,j)| = N(i,j), \ \forall i,j.$$

Therefore, it is inferred that $\bar{G} = \Delta_0 M \Delta_0 \in \mathrm{I}(M)$ and $\bar{H} = \Delta_0 N \Delta_1 \in \mathrm{I}(N)$. From this, and the assumption that Eq. (5) is satisfied we obtain

$$F + \Delta_0(M+M^T)\Delta_0 + \Delta_0 N\Delta_1 Q + Q^T \Delta_1 N^T \Delta_0 = F + \bar{G} + \bar{G}^T + \bar{H}Q + Q^T \bar{H}^T < 0.$$

• Robust constraint (6) implies (5):

Suppose now that robust constraint (6) is satisfied. We proceed by contradiction. If robust constraint (5) is not satisfied, there exists $0 \neq v \in \mathbb{R}^n$ and $\bar{G} \in \mathrm{I}(M)$ and $\bar{H} \in \mathrm{I}(N)$ such that

$$v^T \left(F + \bar{G} + \bar{G}^T + \bar{H}Q + Q^T\bar{H}^T \right) v \geq 0. \quad (7)$$

Now note that

$$\begin{aligned} v^T &\left(F + \bar{G} + \bar{G}^T + \bar{H}Q + Q^T\bar{H}^T \right) v \\ &\leq v^T F v + |v^T \bar{G} v| + |v^T \bar{G}^T v| + |v^T \bar{H} Q v| \\ &\quad + |v^T Q^T \bar{H}^T v| \\ &\leq v^T F v + |v|^T |\bar{G}||v| + |v|^T |\bar{G}|^T |v| \\ &\quad + |v|^T |\bar{H}||Qv| + |Qv|^T |\bar{H}|^T |v| \\ &\leq v^T F v + |v|^T M|v| + |v|^T M^T |v| \\ &\quad + |v|^T N|Qv| + |Qv|^T N^T |v|. \end{aligned}$$

Since the elements of the diagonals of matrices $\boldsymbol{\Delta}_n$ and $\boldsymbol{\Delta}_m$ are equal to 1 or to -1, given v, there exists $\bar{\Delta}_0 \in \boldsymbol{\Delta}_n$ such that $|v| = \bar{\Delta}_0 v$. On the other hand, given Qv there exists $\bar{\Delta}_1 \in \boldsymbol{\Delta}_m$ such that $\bar{\Delta}_1 Qv = |Qv|$. This yields

$$\begin{aligned} v^T F v &+ |v|^T M|v| + |v|^T M^T |v| + |v|^T N|Qv| \\ &\quad + |Qv|^T N^T |v| \\ &= v^T F v + v^T \bar{\Delta}_0 M \bar{\Delta}_0 v + v^T \bar{\Delta}_0 M^T \bar{\Delta}_0 v \\ &\quad + v^T \bar{\Delta}_0 N \bar{\Delta}_1 Q v + v^T Q^T \bar{\Delta}_1 N^T \bar{\Delta}_0 v \\ &= v^T (F + \bar{\Delta}_0 M \bar{\Delta}_0 + \bar{\Delta}_0 M^T \bar{\Delta}_0 \\ &\quad + \bar{\Delta}_0 N \bar{\Delta}_1 Q + Q^T \bar{\Delta}_1 N^T \bar{\Delta}_0) v < 0. \end{aligned}$$

The last inequality is due to the fact that the robust constraint (6) is assumed to be satisfied. Thus, it has been inferred that

$$v^T \left(F + \bar{G} + \bar{G}^T + \bar{H}Q + Q^T\bar{H}^T \right) v < 0$$

which contradicts Eq. (7). ∎

The following corollary states that, in order to check if the robust constraint (5) is satisfied, it suffices to evaluate 2^{n+m} matrices.

Corollary 1. *To check if the robust constraint (5) is satisfied, it suffices to check the matrix inequality (6) for each of the 2^{n+m} different pairs (Δ_0, Δ_1) that satisfy $\Delta_0 \in \boldsymbol{\Delta}_n$, $\Delta_1 \in \boldsymbol{\Delta}_m$.*

Proof. The result stems directly from Theorem 1 and the fact that $\boldsymbol{\Delta}_n$ and $\boldsymbol{\Delta}_m$ have 2^n and 2^m different elements, respectively. ∎

Recall that for small and medium size L_2 gain synthesis problems discussed at the beginning of this section, $2^{19} > 10^5$ and $2^{59} > 10^{17}$ extreme realizations are required to check the robust constraint (5) if full interval matrices are considered. Applying Corollary 1, these numbers drop to $2^9 = 512$ and $2^{16} = 65{,}536$ extreme realizations, respectively. It is clear that in order to solve a semi-definite programming problem with 65,536 constraints, specialized algorithms have to be considered (for example, cutting plane or bundle algorithms). The point here is that the vertex result presented in this section allows one to solve in an exact way (by means of appropriate algorithms) the robust synthesis problem for small and medium size problems. To solve problems of higher dimensions, approximate approaches must be considered. This is due to the inherent NP-hardness nature of this class of robustness problems [12,13]. In Section 7 we present some sufficient conditions that allow one to solve problems of higher dimensions. The price to pay is that the results might be conservative.

6. Generalization to multiaffine interval matrix uncertainty

In this section we show that the results presented in the previous section can be easily generalized to other robustness problems with interval matrix uncertainty. In particular, in the next theorem, we state a vertex result that applies when the uncertain matrices enter in a multiaffine way into the robust constraints. This result in fact generalizes the paper [6] which holds for quadratic stability. Multiaffine uncertainty structures are of interest because they play a major role in control systems [1].

Theorem 2. *Suppose that the matrix function $\Psi(G, H_0, H_1, \ldots, H_p)$ can be rewritten as*

$$\begin{aligned} \Psi&(G, H_0, H_1, \ldots, H_p) \\ &= F_0 + G + G^T + H_0 Q_0 + Q_0^T H_0^T + \prod_{i=1}^{p} (F_i + R_i H_i Q_i) \\ &\quad + \left(\prod_{i=1}^{p} (F_i + R_i H_i Q_i) \right)^T, \end{aligned}$$

for appropriate constant matrices F_i, Q_i, $i = 0, \ldots, p$ and R_i, $i = 1, \ldots, p$. Consider now the robust constraint

$$\Psi(G, H_0, H_1, \ldots, H_p) < 0, \quad \forall G \in I(M), \forall H_i \in I(N_i), \quad i = 0, \ldots, p \quad (8)$$

where $M \in \mathbb{R}^{n_0 \times n_0}$ and $N_i \in \mathbb{R}^{n_i \times m_i}$, $i = 0, \ldots, p$. Then the robust constraint is satisfied if and only if

$$\Psi(\Delta_{l,0} M \Delta_{l,0}, \Delta_{l,0} N_0 \Delta_{r,0}, \ldots, \Delta_{l,p} N_p \Delta_{r,p}) < 0 \quad \forall \Delta_{l,i} \in \boldsymbol{\Delta}_{n_i},\ \Delta_{r,i} \in \boldsymbol{\Delta}_{m_i}\ i = 0, \ldots, p.$$

Proof. From the direct application of Theorem 1, it is inferred that the robust constraint is satisfied if and only if

$$\Psi(\Delta_{l,0} M \Delta_{l,0}, \Delta_{l,0} N_0 \Delta_{r,0}, H_1, \ldots, H_p) < 0 \quad \forall \Delta_{l,0} \in \boldsymbol{\Delta}_{n_0},\ \Delta_{r,0} \in \boldsymbol{\Delta}_{m_0}, \forall H_i \in I(N_i)\ i = 1, \ldots, p.$$

Notice now that given j with $1 \leq j \leq p$,

$$\Psi(\Delta_{l,0} M \Delta_{l,0}, \Delta_{l,0} N_0 \Delta_{r,0}, H_1, \ldots, H_p) = \hat{F}_j + \hat{F}_j^T + \hat{R}_j H_j \hat{Q}_j + \hat{Q}_j^T H_j^T \hat{R}_j^T,$$

where matrices $\hat{F}_j$, $\hat{Q}_j$ and $\hat{R}_j$ do not depend on H_j. Using now the same lines of the proof of Theorem 1, it can be proved that

$$\hat{F}_j + \hat{F}_j^T + \hat{R}_j H_j \hat{Q}_j + \hat{Q}_j^T H_j^T \hat{R}_j^T < 0$$

for all $H_j \in I(N_j)$ if and only if

$$\hat{F}_j + \hat{F}_j^T + \hat{R}_j \Delta_{l,j} N_j \Delta_{r,j} \hat{Q}_j + \hat{Q}_j^T \Delta_{r,j} N_j^T \Delta_{l,j} \hat{R}_j^T < 0$$

for all $\Delta_{l,j} \in \boldsymbol{\Delta}_{n_j}$ and for all $\Delta_{r,j} \in \boldsymbol{\Delta}_{m_j}$. This means that the robust constraint is satisfied for every $H_j \in I(N_j)$ if and only if it is satisfied for every H_j in $\{\Delta_{l,j} N_j \Delta_{r,j} : \Delta_{l,j} \in \boldsymbol{\Delta}_{n_j}, \Delta_{r,j} \in \boldsymbol{\Delta}_{m_j}\}$. This proves the result. ■

This vertex result encompasses a rather large class of robustness problems with multiaffine interval matrix uncertainty. In fact, it can be shown that

$$\prod_{i=1}^{p_a}(\tilde{F}_i + \tilde{R}_i H_i \tilde{Q}_i) + \prod_{i=p_a+1}^{p_a+p_b}(\tilde{F}_i + \tilde{R}_i H_i \tilde{Q}_i)$$

can be rewritten as

$$\prod_{i=1}^{p_a+p_b}(F_i + R_i H_i Q_i)$$

for appropriate matrices F_i, R_i, and Q_i, $i = 1, \ldots, p_a + p_b$.

In order to evaluate the necessary and sufficient conditions stated in Theorem 2, one can take advantage of the structure of the particular robustness problem under study. For instance, in some quadratic stabilization problems the matrices defining the dynamics of the system are in companion form. Therefore, the interval uncertainty does not affect all their components and some rows of the corresponding perturbation scale matrix have their elements equal to zero. In particular, if all the elements in a row of a given perturbation scale matrix are zero, one can use this fact to reduce by a factor of two the number of matrix inequalities required for the robust constraint (see Section 8.3 for an illustrative example). Similar strategies can be adopted in other situations to further reduce this number.

7. Some sufficient conditions

In this section a sufficient condition that avoids the vertex enumeration corresponding to the sets $\boldsymbol{\Delta}_m$ is presented. This sufficient condition takes into account only the 2^n matrices of the set $\boldsymbol{\Delta}_n$. Later, and using this sufficient condition, another sufficient condition that requires no vertex enumeration at all is provided. The technical tool used in this section is the scaling technique presented in [3]. Advantages of the results presented here compared to [3] are discussed at the end of this section.

First, we present a (standard) lemma.

Lemma 1. *Suppose that $S \in \mathbb{R}^{m\times m}$ is a positive definite diagonal matrix. Then*

$$\Delta_0 N \Delta_1 Q + Q^T \Delta_1 N^T \Delta_0 \leq \Delta_0 N S N^T \Delta_0 + Q^T S^{-1} Q,$$
$$\forall \Delta_0 \in \boldsymbol{\Delta}_n, \forall \Delta_1 \in \boldsymbol{\Delta}_m.$$

Proof. We note that $AA^T \geq 0, \forall A$. Thus, we have

$$(\Delta_0 N S^{\frac{1}{2}} - Q^T \Delta_1 S^{-\frac{1}{2}})(\Delta_0 N S^{\frac{1}{2}} - Q^T \Delta_1 S^{-\frac{1}{2}})^T \geq 0.$$

Equivalently,

$$\Delta_0 N S N^T \Delta_0 + Q^T \Delta_1 S^{-1} \Delta_1 Q - \Delta_0 N \Delta_1 Q - Q^T \Delta_1 N^T \Delta_0 \geq 0.$$

That is,

$$\begin{aligned} &\Delta_0 N \Delta_1 Q + Q^T \Delta_1 N^T \Delta_0 \\ &\quad \leq \Delta_0 N S N^T \Delta_0 + Q^T \Delta_1 S^{-1} \Delta_1 Q \\ &\quad = \Delta_0 N S N^T \Delta_0 + Q^T S^{-1} Q. \end{aligned}$$

The last equality is due to the diagonal nature of S^{-1} and the fact that the diagonal elements of $\Delta_1 \in \boldsymbol{\Delta}_m$ satisfy $\Delta_1(i,i)\Delta_1(i,i) = 1, i = 1, \ldots, m$. ■

The following result should be considered as a preliminary result required to prove Theorem 4, which is the main contribution of this section.

Theorem 3. *Consider the robust constraint*

$$F + G + G^T + HQ + Q^T H^T < 0 \quad \forall G \in \mathrm{I}(M), \quad \forall H \in \mathrm{I}(N) \tag{9}$$

where $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times m}$. The robust constraint (9) is satisfied if there exists a diagonal matrix $S \in \mathbb{R}^{m\times m}$ such that, for every $\Delta_0 \in \boldsymbol{\Delta}_n$, we have

$$\begin{bmatrix} F + \Delta_0 \left(M + M^T + N S N^T\right) \Delta_0 & Q^T \\ Q & -S \end{bmatrix} < 0. \tag{10}$$

Proof. From Eq. (10) we conclude that S is a positive definite matrix. Applying Schur complement it results that (10) implies

$$F + \Delta_0 \left(M + M^T + N S N^T\right) \Delta_0 + Q^T S^{-1} Q < 0, \quad \forall \Delta_0 \in \boldsymbol{\Delta}_n.$$

This inequality can be immediately rewritten as

$$F + \Delta_0 (M + M^T) \Delta_0 + \Delta_0 N S N^T \Delta_0 + Q^T S^{-1} Q < 0, \quad \forall \Delta_0 \in \boldsymbol{\Delta}_n.$$

Applying Lemma 1, it follows that:

$$F + \Delta_0 (M + M^T) \Delta_0 + \Delta_0 N \Delta_1 Q + Q^T \Delta_1 N^T \Delta_0 < 0, \quad \forall \Delta_0 \in \boldsymbol{\Delta}_n, \ \forall \Delta_1 \in \boldsymbol{\Delta}_m.$$

As claimed in Theorem 1, this is equivalent to the robust satisfaction of the constraint given by inequality (9). This completes the proof. ■

The next theorem shows how to address (in a conservative way) the robust synthesis problem solving an LMI optimization problem in which $2(n + m)$ scaling variables are added to the original decision variables of the problem.

Theorem 4. *Consider the robust constraint*

$$F + G + G^T + HQ + Q^T H^T < 0 \quad \forall G \in \mathrm{I}(M), \quad \forall H \in \mathrm{I}(N) \tag{11}$$

where $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times m}$. The robust constraint (11) is satisfied if there exist diagonal matrices $T \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{m\times m}$ such that

$$\begin{bmatrix} F + T & Q^T \\ Q & -S \end{bmatrix} < 0, \tag{12}$$

$$M + M^T + N S N^T < T. \tag{13}$$

Proof. First, it is clear that $\Delta_0 T \Delta_0 = T$, therefore, Eq. (12) implies

$$\begin{bmatrix} F + \Delta_0 T \Delta_0 & Q^T \\ Q & -S \end{bmatrix} < 0, \quad \forall \Delta_0 \in \boldsymbol{\Delta}_n.$$

Since $M + M^T + NSN^T < T$, from this equation we conclude that, for every $\Delta_0 \in \boldsymbol{\Delta}_n$

$$\begin{bmatrix} M + \Delta_0 \left(M + M^T + NSN^T \right) \Delta_0 & Q^T \\ Q & -S \end{bmatrix} < 0.$$

As stated in Theorem 3, this last matrix inequality guarantees that the robust constraint is satisfied. This completes the proof. ■

The main advantage of Theorem 4 with respect to the sufficient conditions presented in [3] (Proposition 3.3 in particular) is that the number of additional auxiliary decision variables required in Theorem 4 is much smaller than the number of uncertain parameters. Moreover, the size of the matrices appearing in the obtained LMI is of the same order of magnitude as those corresponding to a nominal synthesis problem. For example, if the results presented in [3] are applied to the robustness problem presented in this paper, then the number of scaling variables would be equal to the number of uncertain parameters n^2+nm and the dimension of the required LMI would be the dimension of the original nominal problem plus n^2+nm. However, it is worth remarking that if the scaling technique of [3] is used to obtain a conservative bound on the largest size of the interval uncertainty for which a given robust LMI (of the class considered in this paper) is feasible, then the obtained upper and lower bounds differ by a factor no larger than $\frac{\pi}{2}$.

8. Numerical results

In this section the theoretical results of this paper are applied to the three motivating examples presented in Section 4. To this end, three different interval systems have been chosen randomly. The numerical results of this section have been obtained with an Intel Pentium 4 at 1.8 GHz using the LMI toolbox of Matlab.

8.1. Quadratic stabilization

Consider the uncertain system $\dot{x} = Ax + Bu$, where $A \in \mathcal{A} = \{A : |A(i,j) - \tilde{A}(i,j)| \le 0.5, \ \forall i, j\}$ and $B \in \mathcal{B} = \{B : |B(i,j) - \tilde{B}(i,j)| \le 0.5, \ \forall i, j\}$ with

$$\tilde{A} = \begin{bmatrix} 0.810 & 7.251 & -4.455 & 9.949 & 9.442 \\ 2.243 & 1.647 & 1.269 & -8.428 & 4.475 \\ -7.743 & -1.024 & -6.522 & 4.892 & 0.711 \\ 9.642 & -5.950 & -8.219 & 9.167 & 9.788 \\ 9.606 & 2.111 & 4.969 & -4.605 & 0.935 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} -8.433 & 9.831 \\ -6.545 & -1.241 \\ -1.954 & -0.264 \\ -4.204 & 8.997 \\ -9.701 & -5.270 \end{bmatrix}.$$

For this interval system, the robust quadratic stabilization problem presented in Section 4.1 can be solved in an exact deterministic way considering only 4,096 elements of the interval family. This is a direct application of Theorem 1. The corresponding generalized eigenvalue problem was solved in 275 seconds yielding the following optimal values for P and K:

$$K = \begin{bmatrix} 25.73 & 148.57 & -73.50 & -89.49 & -45.56 \\ 97.41 & 618.69 & -293.01 & -407.59 & -224.45 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.82 & 4.65 & -2.30 & -2.93 & -1.53 \\ 4.65 & 28.99 & -14.06 & -18.29 & -9.94 \\ -2.30 & -14.06 & 6.99 & 8.60 & 4.59 \\ -2.93 & -18.29 & 8.60 & 12.38 & 6.91 \\ -1.53 & -9.94 & 4.59 & 6.91 & 4.01 \end{bmatrix}.$$

The obtained exact optimal value for α is 7.54. A conservative value of α equal to 6.88 was obtained in 1.63 seconds using Theorem 4.

8.2. L_2 gain minimization

Consider now the robust L_2 gain minimization problem presented in Section 4.2. The values for matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$ are

$$\tilde{A} = \begin{bmatrix} -5.814 & -5.042 & 6.728 & -3.491 & 7.007 \\ -0.886 & 3.687 & 4.486 & 8.265 & 9.696 \\ -3.697 & 4.444 & -8.452 & 3.811 & -1.232 \\ -0.166 & -3.774 & -5.688 & -9.158 & -4.198 \\ 8.933 & -7.369 & -3.917 & 7.432 & 1.949 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} -6.250 & 6.296 & -2.336 & 1.017 & -9.319 \end{bmatrix}^T,$$

$$\tilde{C} = \begin{bmatrix} -2.974 & 3.531 & 0.968 & -0.644 & -6.867 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{D} = \begin{bmatrix} 0 \\ 10 \end{bmatrix},$$

$$\tilde{E} = \begin{bmatrix} 8.185 & 5.415 & -8.490 & -7.554 & 0.644 \end{bmatrix}^T.$$

The perturbation scale matrices R_A, R_B, R_C, R_D and R_E are chosen of appropriate dimensions and with all their elements equal to 0.2. In this case, the number of vertices of the interval family is $2^{47} > 10^{14}$. Theorem 1 guarantees that only $2^{14} =$ 16,384 of these vertices are required to solve in an exact way the robust L_2 gain minimization problem. The corresponding optimization problem was solved in 152 seconds. The obtained optimal value for γ is 5.81 and matrices K and P are

$$K = \begin{bmatrix} 100.80 & 114.59 & -83.90 & 301.85 & 266.55 \end{bmatrix},$$

$$P = \begin{bmatrix} 736.3 & 834.0 & -609.4 & 2197.9 & 1930.8 \\ 834.0 & 1019.7 & -703.0 & 2573.9 & 2255.3 \\ -609.4 & -703.0 & 510.4 & -1837.7 & -1616.5 \\ 2197.9 & 2573.9 & -1837.7 & 6664.0 & 5849.3 \\ 1930.8 & 2255.3 & -1616.5 & 5849.3 & 5154.1 \end{bmatrix}.$$

A conservative value of γ equal to 6.41 was obtained in 0.23 seconds using the sufficient condition provided in Theorem 4.

8.3. Receding horizon control of uncertain discrete-time systems

Consider now the robust receding horizon problem presented in Section 4.3. The numerical values for the matrices $\tilde{A}$ and $\tilde{B}$ are

$$\tilde{A} = \begin{bmatrix} 0.68 & 0.49 & 0.98 & 0.32 \\ 0.59 & 0.90 & 0.18 & 0.47 \\ 0.07 & 0.77 & 0.14 & 0.34 \\ 0.88 & 0.66 & 0.76 & 0.01 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0.34 \\ 0.25 \\ 1.00 \\ 0.91 \end{bmatrix}.$$

The perturbation scale matrices R_A, R_B are chosen of appropriate dimensions and with all their elements equal to 0.05. In this case, the number of vertices of the interval family is 2^{20}. The direct application of Theorem 1 yields that 2^{18} vertices have to be considered. However, since the matrix N (see Eq. (4)) has only five rows different from zero, this number of vertices is further reduced to $2^{10} = 1024$. Given the initial condition $x_0 = [0.3498 \ 0.2781 \ 0.0929 \ 0.2405]^T$, the corresponding optimization problem was solved in 59.44 seconds. The obtained optimal value for γ is 2.8985 and the obtained control action to be applied at sample time $k = 0$ is $u_0 = -0.8181$. A conservative value of γ equal to 3.1321 and the control action $u_0 = -0.8124$ were obtained in 0.6810 seconds using the sufficient condition provided in Theorem 4. As this numerical example illustrates, the sufficient conditions presented in this paper broaden the class of interval systems to which a robust receding horizon control is applicable in real time.

9. Conclusions

In this paper a new vertex result dealing with robust satisfaction of a linear matrix inequality affected by interval uncertainty is presented. The main contribution of the paper is to drastically reduce the number of vertices required to check robust feasibility of a candidate problem solution. This allows us to considerably broaden the family of plants affected by interval uncertainty for which the robust synthesis problem can be solved in a deterministic way. The paper also provides some sufficient conditions that can be used when the size of the interval matrices is large and an exact deterministic solution cannot be obtained using the new vertex result. The results are illustrated by means of three motivating examples and related numerical results.

Acknowledgments

The authors acknowledge MCYT-Spain and the European Commission for funding this work (contracts DPI2005-04568 and DPI2007-66718-C04-01) and the Network of Excellence HYCON (contract 511368). We thank Professors B. Ross Barmish and Yasumasa Fujisaki for pointing out Refs. [15] and [18], respectively. We are also very grateful to Professor Jiri Rohn for his valuable comments.

References

[1] B.R. Barmish, New Tools for Robustness of Linear Systems, MacMillan, New York, 1994.
[2] A. Ben-Tal, A. Nemirovski, Lectures on Modern Convex Optimization, in: Analysis, Algorithms, and Engineering Applications, SIAM, Philadelphia, 2001.
[3] A. Ben-Tal, A. Nemirovski, On tractable approximations of uncertain linear matrix inequalities affected by interval uncertainty, SIAM Journal on Optimization 12 (2002) 811–833.
[4] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in Systems and Control Theory, SIAM, Philadelphia, 1994.
[5] G. Calafiore, B.T. Polyak, Stochastic algorithms for exact and approximate feasibility of robust LMIs, IEEE Transactions on Automatic Control 46 (2001) 1755–1759.
[6] F. Garofalo, G. Celentano, L. Glielmo, Stability robustness of interval matrices via Lyapunov quadratic forms, IEEE Transactions on Automatic Control 38 (1993) 281–284.
[7] D. Hertz, The extreme eigenvalues and stability of real symmetric interval matrices, IEEE Transactions on Automatic Control 37 (1992) 532–535.
[8] H.P. Horisberger, P.R. Belanger, Regulators for linear, time invariant plants with uncertain parameters, IEEE Transactions on Automatic Control 21 (1976) 705–708.
[9] V.L. Kharitonov, Asymptotic stability of an equilibrium position of a family of systems of linear differential equations, Differential'nye Uravneniya 14 (1978) 1483–1485.
[10] M.V. Kothare, V. Balakrishnan, M. Morari, Robust constrained model predictive control using linear matrix inequalities, Automatica 32 (1996) 1361–1379.
[11] D. Liberzon, R. Tempo, Common Lyapunov functions and gradient algorithms, IEEE Transactions on Automatic Control 49 (2004) 990–994.
[12] A. Nemirovskii, Several NP-hard problems arising in robust stability analysis, Mathematics of Control, Signal and Systems 6 (1993) 99–105.
[13] S. Poljak, J. Rohn, Checking robust nonsingularity is NP-hard, Mathematics of Control, Signal and Systems 6 (1993) 1–9.
[14] B.T. Polyak, R. Tempo, Probabilistic robust design with linear quadratic regulators, Systems and Control Letters 43 (2001) 343–353.
[15] J. Rohn, Systems of linear interval equations, Linear Algebra and its Applications 126 (1989) 39–78.
[16] N.P. Seif, S.A. Hussein, A.S. Deif, The interval Sylvester equation, Computing 52 (1994) 233–244.
[17] R. Tempo, G. Calafiore, F. Dabbene, Randomized Algorithms for Analysis and Control of Uncertain Systems, Springer-Verlag, London, 2005.
[18] P.M. Young, Robustness analysis with full-structured uncertainties, Automatica 33 (1997) 2131–2145.
[19] P.M. Young, J.C. Doyle, Properties of the mixed μ problem and its bounds, IEEE Transactions on Automatic Control 41 (1996) 155–159.
[20] K. Zhou, J.C. Doyle, K. Glover, Robust and Optimal Control, Prentice-Hall, Upper Saddle River, 1996.