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Control design with hard/soft performance specifications: a Q-parameter randomization approach

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In this paper, we consider robust controller design problems where the design objectives are divided into two categories: hard specifications and soft specifications. Hard specifications need to be met robustly against uncertainties, and are addressed using a classical $\mathcal{H}_\infty/\mu$-synthesis approach. Soft specifications are given as average performance requirements with respect to uncertainty, and are addressed using a probabilistic design method. The key element in this approach is an algorithm for random generation of stable transfer matrices, with a bound on the $\mathcal{H}_\infty$ norm.

1. Introduction

In the standard robust controller design problem, a nominal plant model is augmented with uncertainty and design objectives masks, to form an extended plant $\mathcal{G}$. The control system is then usually described by means of a feedback connection of the extended plant with a block of structured uncertainties, which may include dynamic as well as static, real and complex blocks (see figure 1). The objective of robust controller design in this setting is to determine a controller K that guarantees closed-loop stability and performance for all $\Delta \in \mathbf{\Delta}_H$, where $\mathbf{\Delta}_H$ is a norm-bounded structured uncertainty set. This problem is usually formulated as the one of finding a controller K such that $J_H(K) < \gamma$, where J_H is some performance index, typically the structured singular value of the closed-loop system.

Two well-known drawbacks of this approach are the computational complexity (μ-synthesis is in general NP-hard), and the difficulty of treating for instance robust $\mathcal{H}_2$, or time-domain design objectives. Also, the complexity of μ-synthesis grows with the number of uncertainty or performance channels (number of structure blocks in Δ) that are included in the extended model.

On the other hand, the probabilistic approach (Khargonekar and Tikku 1996, Tempo *et al.* 1997, Chen and Zhou 1998, Calafiore *et al.* 2000) to robust design is suitable for treating general design objectives, but provides only probabilistic guarantees of performance. A common criticism of this approach is that the robustness of certain closed-loop specifications is 'mandatory' and satisfaction only in probability is not acceptable. For instance, nobody would be willing to fly on a 90% stable airplane, but one could perhaps accept a small probability of degradation of, say, vibration suppression in a certain frequency range.

When used for design, the probabilistic approach usually requires a double randomization: in the space of uncertainties *and* in the space of controller parameters (see, e.g. Vidyasagar 2001). One of the key problems in this respect is how to appropriately select the search space for the randomized controllers. The commonly used approach is to select simple fixed-structure controllers with adjustable parameters and to carry the randomized search over these parameters (Koltchinskii *et al.* 2000). However, this approach poses several problems. For instance, how to select an appropriate controller structure? How to decide about the allowed range of variation of the controller parameters? Moreover, the controller parameterization usually includes destabilizing controllers, so that the possibility of randomly selecting a controller such that the closed loop is unstable is ever present.

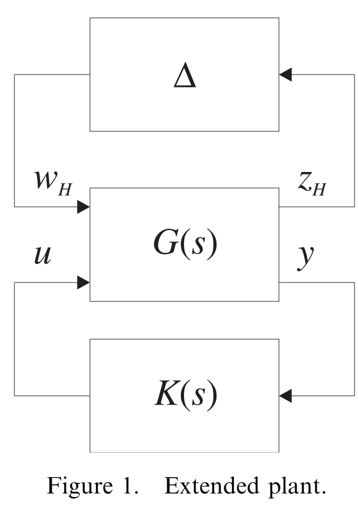

Figure 1. Extended plant.

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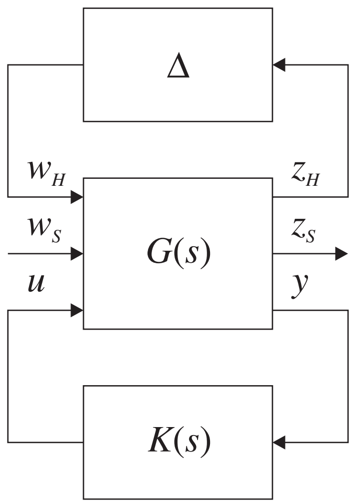

Figure 2. Extended plant with hard and soft performance channels.

In this paper, we propose a probabilistic design technique which overcomes the previously mentioned drawbacks. In particular, we follow a strategy where the randomized controller search is performed using a general parameterization of all stabilizing controllers of fixed-order, thus eliminating the structure restrictions and the possibility of generating destabilizing controllers. Moreover, we take explicit care of the 'mandatory' control requirements, such as robust stability.

The proposed design approach is a mixed deterministic/probabilistic one: we consider two different performance channels $w_H \rightarrow z_H$ and $w_S \rightarrow z_S$, as shown in figure 2, where $w_H \rightarrow z_H$ is the channel related to 'hard' or mandatory performances, that need to be addressed using deterministic robust design methods ($\mathcal{H}_\infty$ or μ-synthesis). The channel $w_S \rightarrow z_S$ represents additional 'soft' performances, which express design objectives that should be achieved in 'average' with respect to a subset $\mathbf{\Delta}_S$ of the uncertainties in $\mathbf{\Delta}_H$.

We proceed as follows. First, a controller is designed using $\mathcal{H}_\infty$ robust design techniques to attain the 'hard' specifications. Then, we consider a standard parameterization $K_Q(s)$ of all robustly stabilizing controllers of fixed-order, in terms of a Q parameter, as discussed in § 2. For every stable $Q(s)$ such that $\|Q(s)\|_\infty < 1$, we have a corresponding fixed-order controller $K_Q(s)$ that attains the hard specifications. At this stage, we proceed with a probabilistic design, performing a double randomization on $\Delta \in \mathbf{\Delta}_S$ and $Q(s)$, in order to achieve the soft specifications, up to a given probabilistic confidence level (see § 3).

The main tools in this latter stage are: (a) the randomization of the uncertainties, which may be performed using the techniques discussed in Calafiore *et al.* (2000) and Calafiore and Dabbene (2002); (b) the randomization on the Q parameter, which is discussed in § 4 of this paper; and (c) the probabilistic evaluation of performance, which is done using the Learning Theory framework proposed in Vidyasagar (1997). A design example illustrating this methodology is presented in § 5.

Notation: Given the configuration in figure 2, we denote by $T_H(s, K)$ the closed-loop transfer matrix between w_H and z_H, when $u = Ky$. We denote by $T_S(s, K, \Delta)$ the closed-loop transfer matrix between w_S and z_S, when $u = Ky$, and $w_H = \Delta z_H$. To indicate that the matrices (A, B, C, D) are a state space realization of a transfer matrix $Q(s)$, we use the standard notation

$$Q(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \doteq C(sI - A)^{-1}B + D.$$

For a square matrix X, $X \succ 0$ (resp. $X \succeq 0$) means X is symmetric and positive-definite (resp. semidefinite). I_n denotes the $n \times n$ identity matrix, while $O_{n,m}$ denotes the $n \times m$ zero matrix. For a real or complex matrix X, $\|X\|_F$ denotes the Frobenius norm of X and $\|X\|$ denotes the spectral (maximum singular value) norm of X. The symbol $\mathcal{RH}_\infty^{n,m}$ denotes the space of stable rational transfer matrices of dimension $n \times m$ (the notation $\mathcal{RH}_\infty$ is also used when the dimensions are left unspecified). The $\mathcal{H}_\infty$ norm of a transfer matrix $Q \in \mathcal{RH}_\infty$ is denoted as $\|Q\|_\infty$.

2. Robust controllers parameterization

The first step of the proposed approach focuses on the 'hard' specifications described in terms of $\mathcal{H}_\infty$ performance. More precisely, for unstructured uncertainty, the controller $K(s)$ has to be designed in order to guarantee that the closed loop satisfies the inequality

$$\|T_H(s, K)\|_\infty < \gamma \tag{1}$$

where γ represents the desired performance level. This corresponds to guaranteeing robust stability for all $\Delta \in \mathbf{\Delta}_H$, where

$$\mathbf{\Delta}_H = \{\Delta \in \mathcal{RH}_\infty \colon \|\Delta\|_\infty \leq 1/\gamma\}.$$

In a more general setting, the uncertainty Δ can be assumed to belong to a structured set of the type

$$\mathcal{D} \doteq \{\Delta \in \mathcal{RH}_\infty \colon \Delta = \text{blockdiag}(\Delta_1, \ldots, \Delta_b, \delta_1 I_{r_1}, \ldots, \delta_s I_{r_s})\} \tag{2}$$

where $\Delta_i \in \mathcal{RH}_\infty$, $i = 1, \ldots, b$ are full blocks, usually related to unmodelled dynamics or design masks, and $\delta_i \in \mathbb{R}$, $i = 1, \ldots, s$ are parametric uncertainties, having multiplicity $r_1, \ldots, r_s$. The robust performance specification then takes the form

$$\sup_\omega \mu(T_H(j\omega, K)) < \gamma \tag{3}$$

where the structured singular value μ is defined as

$$\mu(T_H) \doteq \frac{1}{\min\{\overline{\sigma}(\Delta)\colon \Delta \in \mathcal{D},\ \det(I - T_H\Delta) = 0\}}.$$

If (3) is satisfied, then robust stability is satisfied for all $\Delta \in \mathbf{\Delta}_H$, where

$$\mathbf{\Delta}_H = \{\Delta \in \mathcal{D}\colon \|\Delta\|_\infty \le 1/\gamma\}. \qquad (4)$$

The following well-known lemma provides a parameterization of all possible stabilizing controllers guaranteeing the bound (1) (see, e.g. Zhou *et al.* 1996). For the sake of readability, we limit our discussion to the case when the so-called regularity assumptions hold. The reader interested in the general case may again refer to Zhou *et al.* (1996).

Lemma 1: *Consider a system described as*

$$\mathcal{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \qquad (5)$$

with (A, B_1) *controllable,* (A, B_2) *stabilizable,* (C_1, A) *observable and* (C_2, A) *detectable. Suppose further that the following regularity assumptions hold*

$$D_{12}^{\mathrm{T}}[\,C_1 \quad D_{12}\,] = [\,0 \quad I\,]$$

$$D_{21}[\,B_1^{\mathrm{T}} \quad D_{21}^{\mathrm{T}}\,] = [\,0 \quad I\,].$$

Then the system (5) *admits a stabilizing controller* $K(s)$ *that guarantees a performance level* $\|T_H(s, K)\|_\infty < \gamma$ *if and only if there exist matrices* $X \succ 0$ *and* $Y \succ 0$ *such that*

$$A^{\mathrm{T}}X + XA + X(\gamma^{-2}B_1B_1^{\mathrm{T}} - B_2B_2^{\mathrm{T}})X + C_1^{\mathrm{T}}C_1 = 0 \qquad (6)$$

$$AY + YA^{\mathrm{T}} + Y(\gamma^{-2}C_1^{\mathrm{T}}C_1 - C_2^{\mathrm{T}}C_2)Y + B_1B_1^{\mathrm{T}} = 0 \qquad (7)$$

$$\rho(X, Y) < \gamma^2. \qquad (8)$$

Moreover, the family of all controllers $K_Q(s)$ *guaranteeing* $\|T_H(s, K_Q)\|_\infty < \gamma$ *may be parameterized as*

$$K_Q(s) = \mathcal{F}_\ell(\tilde{K}(s), Q(s)) \qquad (9)$$

where $\mathcal{F}_\ell(\tilde{K}(s), Q(s))$ *denotes the standard lower fractional transformation of K over Q, and where* $Q(s) \in \mathbf{Q}(\gamma)$, $\mathbf{Q}(\gamma) \doteq \{Q(s) \in \mathcal{RH}_\infty\colon \|Q(s)\|_\infty < \gamma\}$ *and* $\tilde{K}(s)$ *is a transfer matrix given by*

$$\tilde{K} = \left[\begin{array}{c|cc} A + (\gamma^{-2}B_1B_1^{\mathrm{T}} - B_2B_2^{\mathrm{T}})X - ZYC_2^{\mathrm{T}}C_2 & ZYC_2 & ZB_2 \\ \hline -B_2^{\mathrm{T}}X & 0 & I \\ -C_2 & I & 0 \end{array}\right] \qquad (10)$$

with $Z \doteq (I - \gamma^{-2}YX)^{-1}$.

The importance of the above lemma is that it explicitly defines the set of all controllers $K_Q(s)$ that achieve the desired closed-loop performance (1), in terms of a single parameter $Q(s) \in \mathbf{Q}(\gamma)$. This parameterization is a standard one, and has been extensively exploited in the literature on deterministic robust control (see, e.g. Zhou *et al.* 1996, Tay *et al.* 1998).

Note that the same kind of parameterization may also be easily determined for problems with structured uncertainty of type (2) (μ-synthesis problems). In fact, if one determines suitable scalings $D(s) \in \mathcal{RH}_\infty$ such that $D^{-1}(s) \in \mathcal{RH}_\infty$, and $D(s)\Delta = \Delta D(s)$ for $\Delta \in \mathcal{D}$ (see, e.g. Fan *et al.* 1991), then any controller $K_Q(s)$ that solves the $\mathcal{H}_\infty$ problem

$$\|DT_H(s, K_Q)D^{-1}\|_\infty < \gamma \qquad (11)$$

also satisfies the μ bound (3). Therefore, it is possible to apply Lemma 1 to the augmented system

$$\tilde{\mathcal{G}} \doteq \mathrm{diag}(D(s), I)\,\mathcal{G}\,\mathrm{diag}(D^{-1}(s), I) \qquad (12)$$

to obtain a family of stabilizing controllers achieving the hard specifications (3). Note however that in the structured case, since (11) is only a sufficient condition for (3), the resulting family of controllers does not contain *all* possible stabilizing controllers satisfying (3). In the sequel of this paper, we focus on the standard $\mathcal{H}_\infty$ design problem of Lemma 1, and therefore assume that

$$J_H(K) = \|T_H(s, K)\|_\infty.$$

The μ-synthesis problem is in fact reduced to the standard $\mathcal{H}_\infty$ problem, if we assume that suitable scalings $D(s)$ are obtained using the D–K iterations method, and the plant is augmented as in (12) (see, e.g. the example in § 5).

3. Design with hard/soft performance specifications

All controllers achieving the desired hard performance J_H are parameterized by the set $\mathbf{Q}(\gamma)$ defined above. We now restrict our attention to *fixed-order* controllers, parameterized by transfer matrices $Q(s) \in \mathbf{Q}(\gamma)$ having fixed McMillan degree n, and denote this set as $\mathbf{Q}_n(\gamma)$

$$\mathbf{Q}_n(\gamma) \doteq \{Q(s) \in \mathcal{RH}_\infty\colon \deg Q(s) = n \text{ and } \|Q(s)\|_\infty \le \gamma\}. \qquad (13)$$

This set is then chosen as the search space for controllers, in order to achieve the so-called *soft* specifications in a probabilistic way.

The soft performance is addressed by considering a finite-dimensional subset $\mathbf{\Delta}_S$ of the uncertainty set $\mathbf{\Delta}_H$. The reason for this is that $\mathbf{\Delta}_H$ may in general contain *fictious* uncertainty blocks that were introduced for the purpose of imposing robust performance specifications in the $\mathcal{H}_\infty$ design. These fictious blocks are neglected when addressing the soft performances, since they do not represent actual uncertainties acting on the plant. For instance, when $\mathbf{\Delta}_H$ is given by (4), one may typically choose $\mathbf{\Delta}_S$ as

$$\mathbf{\Delta}_S = \{\Delta = \operatorname{diag}(0, \ldots, 0, \delta_1 I_{r_1}, \ldots, \delta_s I_{r_s})\colon \|\Delta\| \le 1/\gamma\}$$

where $\delta_i \in \mathbb{R}$, $i = 1, \ldots, s$ are the parametric uncertainty terms.

Formally, given a controller $K_Q(s)$ parameterized by $Q \in \mathbf{Q}_n(\gamma)$, and uncertainty $\Delta \in \mathbf{\Delta}_S \subseteq \mathbf{\Delta}_H$, we define the cost function

$$J_S(Q, \Delta)\colon \{\mathbf{Q}_n(\gamma) \times \mathbf{\Delta}_S\} \to [0, 1] \tag{14}$$

relative to the soft performance channel $T_S(s, K_Q, \Delta)$. For instance, these performance requirements may be expressed as a function of the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the $w_S \to z_S$ channel as

$$J_S(Q, \Delta) = \frac{\|T_S(s, K_Q, \Delta)\|_a}{1 + \|T_S(s, K_Q, \Delta)\|_a}, \quad \text{with } a = 2 \text{ or } \infty. \tag{15}$$

In general, it is also possible to consider other types of specification on the transfer function $T_S(s, K_Q, \Delta)$, such as time-domain specifications with respect to a given input signal ('fuel' consumption, overshoot, ℓ_1 signal norm, etc.).

The performance J_S is not intended to be achieved in a worst-case sense against all $\Delta \in \mathbf{\Delta}_S$, but has to be met in an *average* sense. To this aim, we first need to assume a probability density function (pdf) $f_\Delta(\Delta)$ on the set $\mathbf{\Delta}_S$. This pdf may naturally arise from the plant description (uncertainty resulting from statistical identification, or known distribution of uncertain components in the plant), or can be artificially assumed in a way suitable for control design. For instance, uniform distribution of the uncertainty is often assumed, due to its worst-case properties (see Barmish and Lagoa 1997).

The objective of the controller in this context is to minimize the expected value of J_S with respect to f_Δ, that is the average performance of the controller K_Q when the uncertainty is distributed according to $f_\Delta(\Delta)$. Since the above problem is computationally intractable in general, we seek an approximate solution of the problem

$$\phi = \min_{Q \in \mathbf{Q}_n(\gamma)} E_\Delta[J_S(Q, \Delta)] \tag{16}$$

by means of a randomized algorithm, as proposed in Vidyasagar (2001). This technique is based on randomization over both the uncertainty set $\mathbf{\Delta}_S$ and the controller parameter set $\mathbf{Q}_n(\gamma)$.

First, M independent identically distributed (iid) random samples $Q^1, \ldots, Q^M \in \mathbf{Q}_n(\gamma)$ and N iid samples $\Delta^1, \Delta^2, \ldots, \Delta^N \in \mathbf{\Delta}_S$ are generated. For each Q^k, the *empirical mean* $\hat{E}_N(Q^k)$ (which is an estimate of the 'true' mean $E_\Delta[J_S(Q^k, \Delta)]$) is computed as

$$\hat{E}_N(Q^k) \doteq \frac{1}{N} \sum_{i=1}^{N} J_S(Q^k, \Delta^i). \tag{17}$$

Then, the solution of (16) is approximated by

$$\hat{\phi}_{NM} = \min_{k=1,\ldots,M} \hat{E}_N(Q^k). \tag{18}$$

The following lemma states the probabilistic properties of this solution.

Lemma 2 (Vidyasagar 2001)**:** *Let* $J_S(Q, \Delta)\colon \{\mathbf{Q}_n(\gamma) \times \mathbf{\Delta}_S\} \to [0, 1]$. *Given* $\delta, \epsilon, \alpha \in (0, 1)$, *choose*

$$M \ge \frac{\log(2/\delta)}{\log(1/(1-\alpha))} \quad \text{and} \quad N \ge \frac{\log(4M/\delta)}{2\epsilon^2}.$$

Then, with confidence δ, *we can say that* $\hat{\phi}_{NM}$ *given by* (18) *is an approximate near minimum of* $f(Q) \doteq E_\Delta[J_S(Q, \Delta)]$ *in* (16), *to accuracy* ϵ *and level* α. *That is, the event*

$$\Pr\left\{Q \in \mathbf{Q}_n(\gamma)\colon f(Q) < \hat{\phi}_{NM} - \epsilon\right\} \le \alpha$$

occurs with probability greater than $1 - \delta$.†

By application of the previous lemma, the main result for controller synthesis follows, which is summarized in the next theorem.

Theorem 1: *Let all symbols be defined as in Lemma 1 and Lemma 2, and suppose there exist* $X \succ 0$, $Y \succ 0$ *such that* (6)–(8) *are satisfied. Let further*

$$Q_p \doteq \arg \min_{k=1,\ldots,M} \frac{1}{N} \sum_{i=1}^{N} J_S(Q^k, \Delta^i). \tag{19}$$

Then, the probabilistic controller

$$K_{Q_p}(s) = \mathcal{F}_\ell(\tilde{K}(s), Q_p(s)) \tag{20}$$

is guaranteed to robustly satisfy the hard specification

$$\|T_H(s, K_{Q_p})\|_\infty < \gamma$$

† Note that the whole statement can be compactly written in terms of probabilities as

$$\begin{aligned} &\mathrm{Prob}_{\mathbf{Q}_n(\gamma)^M \times \mathbf{\Delta}_S^N}\{(Q^1 \cdots Q^M) \in \mathbf{Q}_n(\gamma)^M, (\Delta^1 \ldots \Delta^N) \in \mathbf{\Delta}_S^N\colon \\ &\quad \mathrm{Prob}_{\mathbf{Q}_n(\gamma)}\{Q \in \mathbf{Q}_n(\gamma)\colon \\ &\quad f(Q) < \hat{\phi}_{NM} - \epsilon\} \le \alpha\} > 1 - \delta. \end{aligned}$$

and to provide approximately optimal average performance on the soft channel, i.e.

$$\Pr\left\{\hat{\phi}_{NM} \geq \phi + \alpha\right\} \leq \epsilon$$

with probability greater than $1 - \delta$.

Remark 1 (computational complexity)**:** We note that the computational effort necessary to determine the probabilistic controller in (19) is $O(NM\nu)$, where ν is the number of operations required to *evaluate* the performance function J_S for *given* Q^k, Δ^i. Moreover, N, M do not depend on the dimension of the problem (extended plant order and/or number of uncertainty blocks), but only on the desired probabilistic levels δ, ϵ, α. The above complexity figure should actually be multiplied by the number of operations required to generate the random samples Q^k, Δ^i. The complexity of generating Δ^i strongly depends on the choice of the underlying distribution (see Calafiore *et al.* 2000, Calafiore and Dabbene 2002). The complexity of generating Q^k using the algorithm proposed in § 4.1 is $O(n^3)$.

3.1. *Special case: $\mathcal{H}_2$ soft performance*

In this section, we analyse the special case when the performance J_S on the soft channel $w_S \to z_S$ is the normalized $\mathcal{H}_2$ norm, and show the connection between our framework and the deterministic mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design. Consider the system in figure 2, where the hard performance specification is expressed in terms of the $\mathcal{H}_\infty$ norm as

$$J_H(K) = \|T_H(s, K)\|_\infty \leq 1 \tag{21}$$

and therefore $\mathbf{\Delta}_H = \{\Delta \in \mathcal{RH}_\infty\colon \|\Delta\|_\infty \leq 1\}$. Now let $\mathbf{\Delta}_S$ be any finite-dimensional subset of $\mathbf{\Delta}_H$ and assume a probability density $f_\Delta(\Delta)$ over $\mathbf{\Delta}_S$. The soft performance is defined as

$$J_S(Q, \Delta) = \frac{\|T_S(s, K_Q, \Delta)\|_2}{1 + \|T_S(s, K_Q, \Delta)\|_2}, \quad \Delta \in \mathbf{\Delta}_S$$

and our design problem is to minimize with respect to $Q \in \mathbf{Q}_n(\gamma)$ the average with respect to $\Delta \in \mathbf{\Delta}_S$ of $J_S(Q, \Delta)$. Note that any $Q \in \mathbf{Q}_n(\gamma)$ guarantees that the hard performance (21) is satisfied. A deterministic counterpart of this problem is to find a fixed-order controller that minimizes the worst-case (with respect to $\mathbf{\Delta}_S$) $\mathcal{H}_2$ norm of the soft channel, subject to the $\mathcal{H}_\infty$ constraint (21). However, this deterministic approach, known as robust $\mathcal{H}_2$ design (Stoorvogel 1993, Feron 1997), is numerically hard, and no exact efficient solutions are currently known. This further justifies the probabilistic approach that we pursue in this paper. An alternative—and more viable—approach found in the deterministic literature is the so-called mixed $\mathcal{H}_2/\mathcal{H}_\infty$ design, where one aims at minimizing the *nominal* $\mathcal{H}_2$ performance, while satisfying an $\mathcal{H}_\infty$ constraint (Khargonekar and Rotea 1991, Zhu *et al.* 1994, Scherer 1995, Sznaier *et al.* 2000). In our set-up, this corresponds to selecting $\mathbf{\Delta}_S \equiv \emptyset$ (i.e. no randomization on the uncertainty is necessary) and performing a random search over the control parameter Q.

Finally, we stress the fact that the randomized approach presented in this paper is not limited to $\mathcal{H}_2$ performance objectives, but can deal with different performance specifications, both in the frequency and time domain (see, e.g. the example in § 5). Fundamental to our approach is a technique to randomly generate the controller parameter $Q(s)$, which is discussed in the next section.

4. Random transfer matrices in $\mathcal{RH}_\infty$

In this section, we present an efficient algorithm for random generation of stable transfer matrices, with a bound on the $\mathcal{H}_\infty$ norm. Our technique is based on a parameterization of all stable transfer matrices of fixed-order, and relies on a modification of the well-known bounded-real lemma (see, e.g. Scherer 1990), which is recalled below. A different approach for generating uniform transfer functions in $\mathcal{RH}_\infty$ is presented in Lagoa *et al.* (2001): this method is based on finite impulse response approximations, and has the advantage of addressing the issue of uniformity of the samples, but appears to be currently limited to the single-input–single-output case.

Lemma 3 (bounded-real lemma)**:** *Let $\gamma > 0$ be given. The following two statements are equivalent:*

(1) $Q(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathcal{RH}_\infty$ *and* $\|Q(s)\|_\infty < \gamma$.

(2) *There exist $P \succ 0$ such that the following LMI in the matrix variable P is satisfied*

$$\begin{bmatrix} PA + A^{\mathrm{T}}P & PB & C^{\mathrm{T}} \\ B^{\mathrm{T}}P & -\gamma I & D^{\mathrm{T}} \\ C & D & -\gamma I \end{bmatrix} \prec 0. \tag{22}$$

We now state the following key corollary.

Corollary 1: *Let $\gamma > 0$ be given, and let $Q(s)$ be a rational and proper transfer matrix with n_i inputs, n_o outputs, and McMillan degree n. The following two statements are equivalent:*

(1) $Q(s) \in \mathbf{Q}_n(\gamma)$.

(2) *There exist a minimal realization A, B, C, D of $Q(s)$, with* $A \in \mathbb{R}^n$, $B \in \mathbb{R}^{n, n_i}$, $C \in \mathbb{R}^{n_o, n}$,

$D \in \mathbb{R}^{n_o, n_i}$, such that the following LMI in the matrix variables A, B, C, D is satisfied

$$\begin{bmatrix} A + A^{\mathrm{T}} & B & C^{\mathrm{T}} \\ B^{\mathrm{T}} & -\gamma I & D^{\mathrm{T}} \\ C & D & -\gamma I \end{bmatrix} \prec 0. \tag{23}$$

Proof: The implication from (2) to (1) is immediately proved, using Lemma 3, with $P = I$. To prove the implication from (1) to (2), consider a minimal realization of $Q(s)$: $\bar{A}, \bar{B}, \bar{C}, \bar{D}$, then by Lemma 3 there exist a $P \succ 0$ such that

$$\begin{bmatrix} P\bar{A} + \bar{A}^{\mathrm{T}} P & P\bar{B} & \bar{C}^{\mathrm{T}} \\ \bar{B}^{\mathrm{T}} P & -\gamma I & \bar{D}^{\mathrm{T}} \\ \bar{C} & \bar{D} & -\gamma I \end{bmatrix} \prec 0. \tag{24}$$

Let P be factored as $P = R^{\mathrm{T}} R$, where R is non-singular, and consider the congruence transformation obtained multiplying (24) by $\mathrm{diag}(R^{-\mathrm{T}}, I, I)$ on the left and $\mathrm{diag}(R^{-1}, I, I)$ on the right. The LMI (24) holds if and only if the following one holds

$$\begin{bmatrix} R\bar{A}R^{-1} + R^{-\mathrm{T}}\bar{A}^{\mathrm{T}}R^{\mathrm{T}} & R\bar{B} & R^{-\mathrm{T}}\bar{C}^{\mathrm{T}} \\ \bar{B}^{\mathrm{T}}R^{\mathrm{T}} & -\gamma I & \bar{D}^{\mathrm{T}} \\ \bar{C}R^{-1} & \bar{D} & -\gamma I \end{bmatrix} \prec 0.$$

The statement is then proved taking

$$\begin{aligned} A &= R\bar{A}R^{-1} \\ B &= R\bar{B} \\ C &= \bar{C}R^{-1} \\ D &= \bar{D}. \end{aligned}$$

That is, the realization A, B, C, D is obtained from $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ by means of a suitable similarity transformation. □

Remark 2: Note that the previous corollary states that *every* stable transfer matrix $Q(s)$ of fixed-order, with $\mathcal{H}_\infty$ norm bound γ, admits a state-space realization which belongs to the feasible set of the LMI (23). In other words, this feasible set represents a convex parameterization of all fixed-order, stable transfer matrices bounded in the $\mathcal{H}_\infty$ norm.

In this setting, we may also deal with further 'realistic' constraints on the transfer matrix. These additional constraints come from practical considerations (e.g. the entries of the realization should not be 'too large'), since the constraint (23) alone would allow for arbitrarily large entries in A, B, C (for instance, if a quadruple (A, B, C, D) is feasible for (23), then any quadruple $(rA, \sqrt{r}B, \sqrt{r}C, D)$ would still be feasible, for any $r > 0$). The next definition describes precisely the class of transfer matrices we shall be concerned with in the sequel.

Definition 1: Let $\lambda, \gamma \succ 0$ be given, and let $A \in \mathbb{R}^{n}$, $B \in \mathbb{R}^{n, n_i}$, $C \in \mathbb{R}^{n_o, n}$, $D \in \mathbb{R}^{n_o, n_i}$. We define as $\mathbf{Q}_n^{\lambda}(\gamma)$ the set of transfer matrices $Q(s) \in \mathbf{Q}_n(\gamma)$ that admit a realization

$$Q(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

which satisfies $\|A\|_F \leq \lambda$ and the matrix inequality (23).

Note that each $Q(s) \in \mathbf{Q}_n^{\lambda}(\gamma)$ is stable, has $\mathcal{H}_\infty$ norm less than γ, and the modula of its poles are bounded by λ, since the bound $\|A\|_F \leq \lambda$ implies that the spectral radius of A (maximum magnitude of the eigenvalues) is no larger than λ. Note also that all the results stated in §4, which considered $\mathbf{Q}_n(\gamma)$ as the search set, also hold if $\mathbf{Q}_n(\gamma)$ is replaced by its subset $\mathbf{Q}_n^{\lambda}(\gamma)$. In the next section we present an algorithm for random generation of transfer matrices in the set $\mathbf{Q}_n^{\lambda}(\gamma)$.

4.1. *Random generation algorithm*

Following Definition 1, we introduce the set

$$\mathcal{X} \doteq \left\{ (A, B, C, D)\colon Q(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbf{Q}_n^{\lambda}(\gamma) \right\} \tag{25}$$

and generate random quadruples (A, B, C, D) in $\mathcal{X}$. The algorithm we propose is divided in three main steps. In the first step, we randomly generate a (Hurwitz) stable system matrix A, which satisfies the required norm bound. Next, we generate the D matrix, uniformly in the spectral norm ball $\|D\| < \gamma$, or set $D = 0$, if strictly proper transfer matrices are required. In the third step, we randomly select the B and C matrices in the feasible set of the LMI (23), A and D being fixed. We analyze these three steps in the following subsections, and next explicitly report the generation algorithm, using pseudo-Matlab code notation when needed.

4.1.1. *Generation of Hurwitz matrix A.*

Define

$$Y \doteq \begin{bmatrix} B & C^{\mathrm{T}} \end{bmatrix}$$

$$R_D \doteq \begin{bmatrix} -\gamma I & D^{\mathrm{T}} \\ D & -\gamma I \end{bmatrix}.$$

Then, by the Schur complements rule, the main LMI (23) holds if and only if

$$A + A^{\mathrm{T}} = -\alpha Z \tag{26}$$

$$R_D + \frac{1}{\alpha} Y^{\mathrm{T}} Z^{-1} Y \prec 0 \tag{27}$$

hold for some $\alpha > 0$, and $Z \succ 0$ such that $\|Z\|_F = 1$. Note that, for given $Z \succ 0$, $\alpha > 0$ all matrices A that solve (26) can be parameterized as

$$A = -\frac{1}{2}\alpha Z + \beta X \tag{28}$$

where $\beta > 0$ and X is any skew-symmetric matrix, normalized so that $\|X\|_F = 1$. Therefore, the idea is to generate A by picking uniformly at random Z and X. To generate a sample of Z, we first write $Z = ULU^{\mathrm{T}}$, where U is orthogonal, and $L = \mathrm{diag}(\ell_1, \ldots, \ell_n) \succ 0$ is normalized so that $\|L\|_F = 1$. Then, a sample of Z can be obtained generating the random orthogonal matrix U uniformly over the orthogonal group (this can essentially be done via a QR factorization of a Normal matrix, see Stewart (1980) and Calafiore *et al.* (2000) for details), and the vector $\ell = [\ell_1, \ldots, \ell_n]$ of eigenvalues uniformly over the intersection of the surface of the unit ball and the positive orthant (such a sample is easily obtained taking $[\ell_1, \ldots, \ell_n]^{\mathrm{T}} = \xi^+/\|\xi\|$, where $\xi^+ \in \mathbb{R}^n$ is the projection on the positive orthant of a Normal vector ξ having zero mean and unit variance). Similarly, a sample of the free term X is drawn uniformly over the surface of the unit ball $\{X = -X^{\mathrm{T}}\colon \|X\|_F \le 1\}$. The scalar parameters α, β are then chosen in order to impose the additional constraint $\|A\|_F \le \lambda$. To this end note that from (28) we have

$$\|A\|_F^2 = \frac{\alpha^2}{4}\|Z\|_F^2 + \beta^2\|X\|_F^2 = \frac{\alpha^2}{4} + \beta^2$$

and therefore if we selected $\alpha^2/4 = \lambda^2 u$, $\beta^2 = \lambda^2(1-u)$, where u is a random scalar uniformly drawn from the interval $[0,1]$, we would obtain a sample of A lying on the surface of the ball $\{A\colon \|A\|_F \le \lambda\}$. Finally, this sample is smudged inside the volume of the ball, by multiplying it by a volumetric factor w^{1/n^2} (see, e.g. Calafiore *et al.* 2000), where w is uniform in $[0,1]$. In conclusion, the parameters α, β are selected as

$$\alpha = 2\lambda w^{1/n^2}\sqrt{u}, \qquad \beta = \lambda w^{1/n^2}\sqrt{1-u}.$$

4.1.2. *Generation of B,C,D.* In the second step of the algorithm, the D term is generated uniformly in the spectral norm ball $\{D\colon \|D\| \le 1\}$, which can be accomplished using the techniques described in Calafiore and Dabbene (2002). Finally, once A and D have been obtained, in the third step we need to generate $Y \doteq [B\ \ C^{\mathrm{T}}] \in \mathbb{R}^{n, n_i+n_o}$ such that (27) is satisfied. To this end, we first generate a random normalized direction $\tilde{Y} \in \mathbb{R}^{n, n_i+n_o}$ (this is again accomplished by drawing a Normal matrix sample of the required dimensions and dividing it by its Frobenius norm), and then consider $Y = r\tilde{Y}$, for some $r > 0$ to be assigned. Note that the set of Ys that are feasible for (27) is convex and centrally symmetric. We therefore proceed by first determining a value of r such that $r\tilde{Y}$ is on the boundary of this feasible set, and then scale the sample inside the feasible set, by multiplying it by its volumetric factor. In particular, if we factor $R_D = -\Gamma\Gamma$, for $\Gamma \succ 0$,† and define $W = (1/\alpha)\Gamma^{-1}(\tilde{Y}^{\mathrm{T}}Z^{-1}\tilde{Y})\Gamma^{-1}$, we have that the value

$$\tilde{r} = \frac{1}{\sqrt{\lambda_{\max}(W)}}$$

is such that $\tilde{r}\tilde{Y}$ is on the boundary of the feasible set. Finally, we determine a point in the interior by multiplying $\tilde{r}\tilde{Y}$ by a volumetric factor $z = w^{1/(n(n_i+n_o))}$, where w is uniform in $[0,1]$, thus obtaining the final sample $Y = r\tilde{Y}$, with $r = z\tilde{r}$.

The complete algorithm is outlined below.

Algorithm 1: Given $n, n_i, n_o, \gamma, \lambda > 0$, with probability one returns an $n_o \times n_i$ random transfer matrix $Q(s) \in \mathbf{Q}_n^{\lambda}(\gamma)$.

Step 1. **Generation of** A

(a) Generate a random orthogonal $U \in \mathbb{R}^{n,n}$;

(b) Generate $\xi = \mathtt{abs}(\mathtt{randn}(n, 1))$, set $\ell = \xi/\mathtt{norm}(\xi)$, and compute $L = \mathtt{diag}(\ell)$, $Z = ULU^{\mathrm{T}}$, $Z^{-1} = UL^{-1}U^{\mathrm{T}}$;

(c) Generate $\xi = \mathtt{randn}(n(n-1)/2, 1)$, and build a skew-symmetric matrix X from the elements in vector x. Normalize $X = X/\|X\|_F$;

(d) Generate $u = \mathtt{rand}$, $w = \mathtt{rand}$, and compute

$$\alpha = 2\lambda w^{1/n^2}\sqrt{u}, \qquad \beta = \lambda w^{1/n^2}\sqrt{1-u};$$

(e) Compute $A = -\frac{1}{2}\alpha Z + \beta X$.

Step 2. **Generation of** D

(a) If strictly proper transfer matrix is required, set $D = 0$;

(b) Else, generate a random matrix $D \in \mathbb{R}^{n_o, n_i}$, uniformly in the spectral norm ball $\|D\| < \gamma$.

Step 3. **Generation of** $Y \doteq [B\ \ C^{\mathrm{T}}]$

(a) Generate $\tilde{Y} = \mathtt{randn}(n, n_i + n_o)$, and normalize $\tilde{Y} = \tilde{Y}/\|\tilde{Y}\|_F$;

(b) Compute symmetric factorization $-R_D = \Gamma\Gamma$, and define $W = (1/\alpha)\Gamma^{-1}(\tilde{Y}^{\mathrm{T}}Z^{-1}\tilde{Y})\Gamma^{-1}$;

(c) Generate $w = \mathtt{rand}$ and compute $z = w^{1/(n(n_i+n_o))}$;

(d) Compute $r = z/\sqrt{\lambda_{\max}(W)}$;

(e) Compute $Y = r\tilde{Y}$.

Step 4. **Return** $Q(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$.

† Note that in the case of strictly proper transfer matrices, we have $D = 0$, and hence Γ is simply given by $\Gamma = \sqrt{\gamma}I$.

Note that the proposed algorithm is not restricted to SISO transfer functions, but works with generic MIMO transfer matrices.

5. Example

We considered a classical two masses spring-dashpot system, with parametric uncertainty on the mass and stiffness values. The control input u is represented by a force acting on the first mass, while the measured output y is the position of the second mass. The extended model given below represents the nominal plant, augmented with design specification masks.

$$\begin{bmatrix} z_1 \\ z_2 \\ z_S \\ y \end{bmatrix} = \left[\begin{array}{ccccc|ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -0.001 & 0.001 & 1 & 0 & 0.8199 & 0 & 0 & 0.8199 & 0.8199 & 0.8199 & 0 \\ 1 & 0.001 & -0.002 & -2 & 0 & 0 & 0 & 0.8199 & 0 & -0.8199 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -400 & 0 & 0.8199 & 0 & 0 & 0 & 0 & 0 \\ \hline 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0.2 & 0.0002 & -0.0004 & -0.4 & 0 & 0 & 0 & 0.164 & 0 & -0.164 & 0 & 0.2 \\ -0.2 & -0.0002 & 0.0002 & 0.2 & 0 & 0.164 & 0 & 0 & 0.164 & 0.164 & 0.164 & 0 \\ -0.2 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -396 & 0 & 0.8199 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \times \begin{bmatrix} w_1 \\ w_2 \\ w_S \\ u \end{bmatrix}.$$

The channel $w_1 \in \mathbb{R}^2 \to z_1 \in \mathbb{R}^2$ takes into account specifications on the sensitivity and command activity, while the channel $w_2 \in \mathbb{R}^3 \to z_2 \in \mathbb{R}^3$ refers to parametric uncertainty. Here, $w_H = [w_1^{\mathrm{T}}\ w_2^{\mathrm{T}}]^{\mathrm{T}} \to z_H = [z_1^{\mathrm{T}}\ z_2^{\mathrm{T}}]^{\mathrm{T}}$ represents the hard specifications channel, while $w_S \in \mathbb{R} \to z_S \in \mathbb{R}$ represents the soft channel, which in this case is the output response to a force disturbance acting on the second mass. In the present situation, the 'uncertainty' acting on the system is represented by the feedback connection

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \mathrm{diag}(\Delta_1, \Delta_2, \delta_1, \delta_2, \delta_3) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where Δ_1, Δ_2 represent two fictious complex scalar blocks related to the design specifications, while $\delta_1, \delta_2, \delta_3$ are true parametric uncertainties on the first and second mass, and on the stiffness, respectively. In this setting, we therefore have

$$\boldsymbol{\Delta}_H = \{\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \delta_1, \delta_2, \delta_3), \Delta_1, \Delta_2 \in \mathbb{C};$$
$$\delta_1, \delta_2, \delta_3 \in \mathbb{R}\colon \|\Delta\| \le 1\}.$$

In order to take this uncertainty structure into account in the design phase, we used $D-K$ iterations of μ synthesis to determine suitable scalings. In the present example, the following first-order scalings were determined

$$D(s) = \mathrm{diag}\left(1.216, 0.0141, \frac{0.02671s + 1.009}{s + 0.000449}, \frac{0.07547s + 1.017}{s + 8.556 \times 10^{-5}}, 1\right).$$

An $\mathcal{H}_\infty$ design was then performed on the scaled plant

$$\tilde{\mathcal{G}} \doteq \mathrm{diag}(D(s), 1)\, \mathcal{G}\, \mathrm{diag}(D^{-1}(s), 1)$$

obtaining a central controller that guarantees

$$\|DT_H(s, K)D^{-1}\|_\infty < 1.$$

At this point, we parameterize the family K_Q of controller solving the above $\mathcal{H}_\infty$ problem using Lemma 1, and apply the proposed randomized method to optimize the average (with respect to the parametric uncertainty) behaviour of the impulse response on the soft channel. To this end, we neglect the fictious uncertainty blocks Δ_1, Δ_2, and consider $\mathbf{\Delta}_S$ as the following subset of $\mathbf{\Delta}_H$

$$\mathbf{\Delta}_S = \{\Delta = \mathrm{diag}(0, 0, \delta_1, \delta_2, \delta_3), \delta_1, \delta_2, \delta_3 \in \mathbb{R}\colon \|\Delta\| \le 1\}.$$

To optimize the time response, we choose the ℓ_1 criterion

$$\tilde{J}_S = \int_0^\infty |z_S(t)|\,\mathrm{d}t$$

where $z_S(t)$ is the response to an impulse on the input w_S, and normalize the objective as $J_S = \tilde{J}_S/(1+\tilde{J}_S)$. Assuming uniform probability distribution on $\mathbf{\Delta}_S$, we optimized the average of J_S with respect to uncertainty realizations. We considered Q parameters of order $n=3$, with spectral radius bound $\lambda = 100$. The randomized algorithm, with $M = 50$, $N = 500$ (corresponding to $\alpha = \epsilon = 0.1$, $\delta = 0.01$ in Lemma 2), yielded a controller which guarantees the hard specifications $\|T_H\|_\infty < 1$, and which provides the optimized time responses on the soft channel depicted in figure 3. These *a posteriori* simulations showed that the randomized controller provided a 25% improvement in the average soft performance with respect to the central controller.

One advantage of the randomized approach is its flexibility, in that it can easily manage time domain specifications (such as the ℓ_1 norm above) that would be otherwise extremely difficult to tackle in a deterministic setting. As a further example, we considered a soft criterion based on the (squared) ℓ_2 norm of the impulse response $z_S(t)$

$$\tilde{J}_S = \int_0^\infty z_S^2(t)\,\mathrm{d}t.$$

The randomized controller has been re-computed using this new criterion, and its performances are compared to those of the central controller in figure 4. In this case, the *a posteriori* simulations showed that the randomized controller provided a 45% improvement in the squared ℓ_2 average performance with respect to the central controller.

6. Conclusions

In this paper, we presented a mixed deterministic/probabilistic approach for robust controller design. The key feature of this method is that the search for probabilistic performance on the 'soft' channel is done while maintaining deterministic robust performance on the 'hard' channel. In the authors' view this approach overcomes many of the current limitations of the available probabilistic design techniques, by blending together the classical robust design methods with the newly emerged randomized techniques.

We remark that the tools presented here may also be used to solve μ-synthesis problems in a probabilistic setting: for an n-blocks μ problem, we first solve an $\mathcal{H}_\infty$

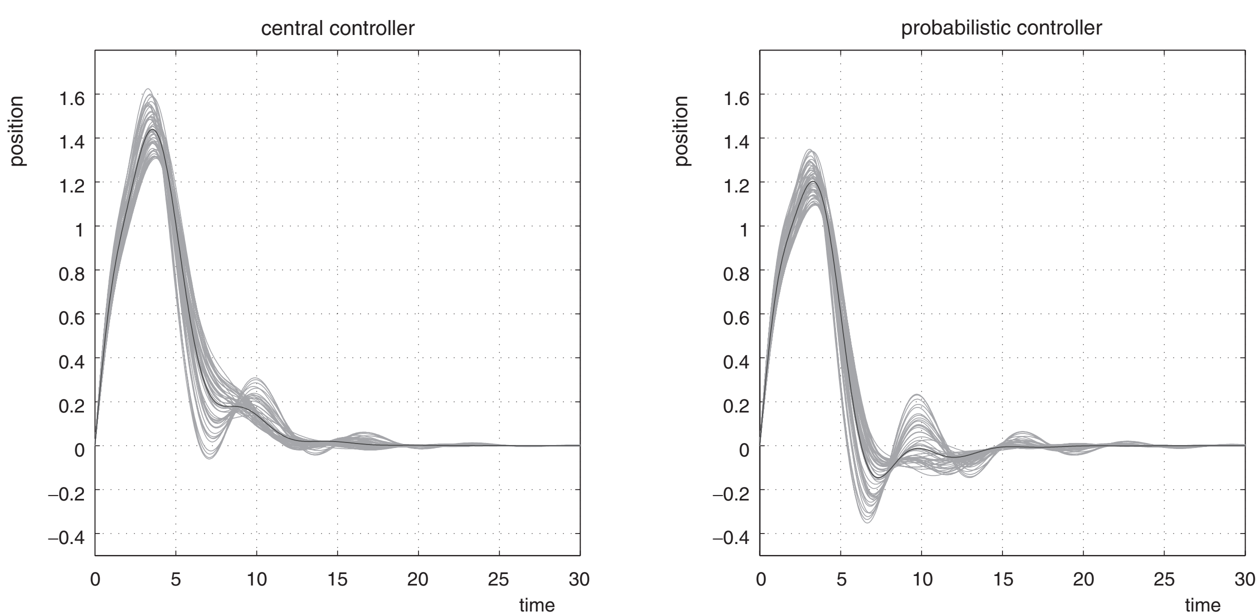

Figure 3. Impulse responses on the soft channel for randomly chosen values of the uncertainty. Central controller (left) and randomized controller (right). The randomized controller is optimized for the average ℓ_1 norm of the response. Darker lines represent the nominal responses (i.e. with Δ set to zero).

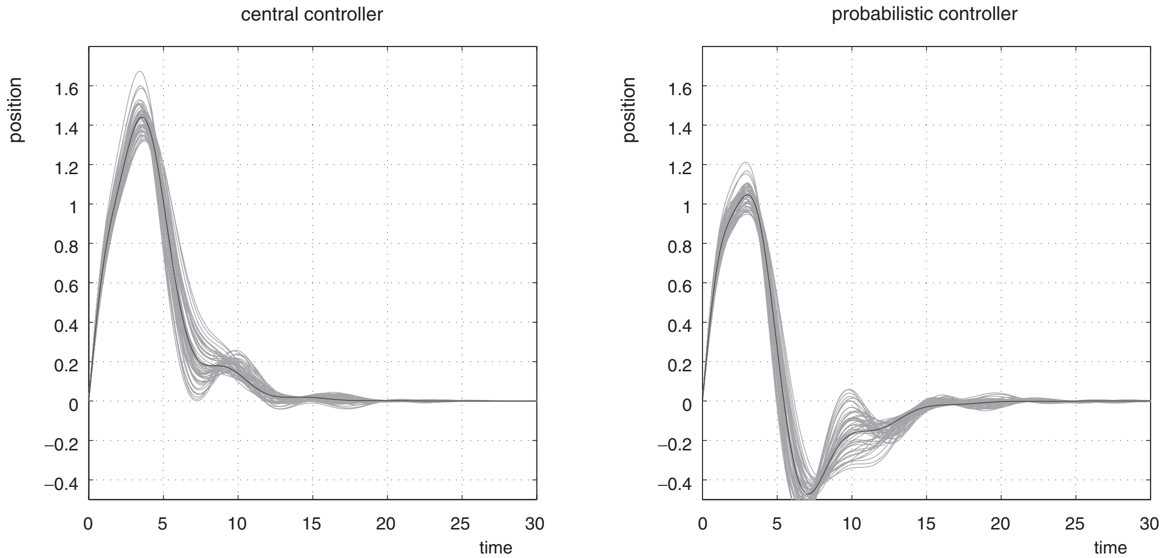

Figure 4. Impulse responses on the soft channel for randomly chosen values of the uncertainty. Central controller (left) and randomized controller (right). The randomized controller is optimized for the average squared ℓ_2 norm of the response. Darker lines represent the nominal responses (i.e. with Δ set to zero).

synthesis problem for, say, the first block alone. Then, we parameterize all such controllers, and note that the controller for the initial n-blocks problem, if it exists, must be in the parameterized family. We may then use a randomized approach to search in this parameterized family for a controller achieving the n-blocks robust design.

The presented method relies on a technique for randomization of fixed-order random matrices in $\mathcal{RH}_\infty$, which is described in §4. Similar results and an analogous algorithm may be easily derived also for the case of discrete-time systems.

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