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RESEARCH PAPER

Optimization under uncertainty with applications to design of truss structures

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Abstract Many real-world engineering design problems are naturally cast in the form of optimization programs with uncertainty-contaminated data. In this context, a reliable design must be able to cope in some way with the presence of uncertainty. In this paper, we consider two standard philosophies for finding optimal solutions for uncertain convex optimization problems. In the first approach, classical in the stochastic optimization literature, the optimal design should minimize the expected value of the objective function with respect to uncertainty (*average* approach), while in the second one it should minimize the worst-case objective (*worst-case* or min–max approach). Both approaches are briefly reviewed in this paper and are shown to lead to exact and numerically efficient solution schemes when the uncertainty enters the data in simple form. For general uncertainty dependence however, these problems are numerically hard. In this paper, we present two techniques based on uncertainty randomization that permit to solve efficiently some suitable probabilistic relaxation of the indicated problems, with full generality with respect to the way in which the uncertainty enters the problem data. In the specific context of truss topology design, uncertainty in the problem arises, for instance, from imprecise knowledge of material characteristics and/or loading configurations. In this paper, we show how reliable structural design can be obtained using the proposed techniques based on the interplay of convex optimization and randomization.

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1 Introduction

A standard convex optimization program is usually formulated as the problem of minimizing a convex cost function $f(x)$ over a convex set $\mathcal{X}$, i.e.,

$$\min_{x \in \mathcal{X}} f(x).$$

In practice, however, the objective function f itself might be imprecisely known. We account for this situation by assuming that function f depends not only on the decision vector x but also on a vector of uncertain parameters δ that are assumed to be random with known distribution over a compact set $\Delta \subset \mathbb{R}^{\ell}$. In this setting, the formulation of the optimization problem and the meaning of solution need be clarified. In fact, it is possible to devise different paradigms that consider the effect of the uncertainty from different viewpoints. One may consider the problem from a min–max viewpoint and look for a solution that minimizes the "worst-case" value (with respect to the uncertainty) of the objective function. Alternatively, one may be interested in considering the effect of the uncertainty "on average," and this corresponds to minimizing an uncertainty-averaged version of the cost function. In this paper, we present these two standard philosophies, and we discuss recent techniques based either on exact or probabilistically approximate solutions. Exact solutions are obtained when the uncertainty enters the objective function in some simply structured form and are discussed in Section 2.1. In the general case, exact solutions are numerically hard to compute. Hence, we present in Section 2.2 algorithms based on uncertainty randomization for efficient solution of probabilistic relaxations of convex problems with generic uncertainty dependence. We refer the reader to Tempo et al. (2004) and Calafiore and Dabbene (2006) for an overview of randomized algorithms and their application to robustness problems and for pointers to the related literature.

Section 4 describes an application of the discussed techniques to problems related to design of mechanical truss structures under uncertainty on the loading patterns and material characteristics.

2 Worst-case and average designs

We formally state the two classes of optimization problems that are the object of our study.

Consider an objective function $f(x,\delta) : \mathcal{X} \times \Delta \to \mathbb{R}$, where $\mathcal{X} \subseteq \mathbb{R}^n$ is a convex set and $\Delta \subset \mathbb{R}^\ell$ is a compact set. Let $\mathbb{P}$ denote a probability measure on δ.

We make the standing assumption that the objective is convex in the decision variables for any fixed value of the uncertainty:

Assumption 1 *The function $f(x,\delta)$ is convex in x for every $\delta \in \Delta$.*

2.1 Worst-case approach

In the worst-case optimization approach, one seeks a solution guaranteed for all possible values taken by the uncertainty $\delta \in \Delta$. This leads to the following *min–max* optimization problem

$$\min_{x\in\mathcal{X}} \max_{\delta\in\Delta} f(x,\delta). \tag{1}$$

When Δ is convex and $f(x,\delta)$ is concave in δ, problem (1) is a saddle-point optimization problem that can be solved, for instance, by means of cutting-plane techniques (see, e.g., Atkinson and Vaidya 1995; Goffin and Vial 2002). Problem (1) can also be restated in an epigraphic form as a robust, or semi-infinite, optimization problem:

$$\min_{t,x\in\mathcal{X}} t \quad \text{subject to}$$
$$f(x,\delta) \le t, \quad \forall \delta \in \Delta. \tag{2}$$

An important class of such problems involves symmetric linear matrix functions of x, with f being the largest eigenvalue function:

$$f(x,\delta) = \lambda_{\max}(F(x,\delta)),$$
$$F(x,\delta) = F_0(\delta) + \sum_{i=1}^n x_i F_i(\delta),$$

where $F_i(\delta)$ are symmetric matrices that depend in a possibly nonlinear way on the uncertainties δ. The constraint $\lambda_{\max}(F(x,\delta)) \le 0$ is equivalent to requiring that $F(x,\delta)$ is negative semidefinite, a condition that we indicate with the notation $F(x,\delta) \preceq 0$. For fixed δ, this is a linear matrix inequality (LMI) condition on x. For examples of different problems that can be cast in LMI form and for additional details on the subject, the interested reader is referred to Boyd et al. (1994) and Lobo et al. (1998). Minimizing a linear objective subject to an LMI constraint is a convex problem [usually referred to as semidefinite optimization program (SDP)] that can be solved very efficiently by means, for instance, of interior point methods (see Todd 2001; Vandenberghe and Boyd 1996). In our context, however, we are interested in problems that are subject to an *infinite* number of LMI constraints, that is in problems of the form

$$\min_{x\in\mathbb{R}^n} c^T x \quad \text{subject to}$$
$$F(x,\delta) \preceq 0, \quad \forall \delta \in \Delta. \tag{3}$$

These problems are called *robust* SDPs in El Ghaoui et al. (1998) and are, in general, very hard to solve numerically (see, e.g., Ben-Tal and Nemirovski 1998, 2002). Efficiently computable and exact solutions are available only when the uncertainty enters the function F in a simple form (such as affine; see, for instance, Ben-Tal and Nemirovski 1998, 2002; El Ghaoui et al. 1998).

We next recall two key results that permit to recast exactly the semi-infinite optimization problem (3) in the form of a standard semidefinite program.

Lemma 1 (Polytopic SDP) *Let $F(x,\delta) \preceq 0$ be an LMI in $x \in \mathbb{R}^n$, parameterized in $\delta \in \mathbb{R}^p$, which is written in standard form as*

$$F(x,\delta) \doteq F_0(\delta) + \sum_{i=1}^n x_i F_i(\delta) \preceq 0,$$

where $F_i(\delta) = F_i^T(\delta)$ are affine functions of δ. Let $\delta \in \Delta$, where Δ is a polytope of vertices $\delta_1, \dots, \delta_v$. Then

$$F(x,\delta) \preceq 0,\ \forall \delta \in \Delta \quad \Leftrightarrow \quad F(x,\delta_i) \preceq 0,\ i = 1,\dots,v. \tag{4}$$

A proof of this lemma is given in the Appendix. When Δ is a polytope and δ enters $F(x,\delta)$ affinely, the previous result permits to substitute the semi-infinite constraint $F(x,\delta) \preceq 0, \forall\Delta \in \Delta$ with a *finite number* of LMI constraints, thus transforming problem (3) into a standard and efficiently solvable SDP.

Along the same line, the next lemma provides a way to recast a robust SDP into a standard one, when F is affected additively by a norm-bounded uncertainty (see El Ghaoui et al. 1998).

Lemma 2 (Norm-bounded SDP) *Let $F(x,\delta)$ have the form*

$$F(x,\delta) = \mathcal{F}(x) + \mathcal{L}\delta\mathcal{R}(x) + \mathcal{R}^T(x)\delta^T\mathcal{L}^T,$$

where $\mathcal{F}(x)$ is an affine, symmetric matrix function of x, $\mathcal{L}$ is a constant matrix, $\mathcal{R}(x)$ is affine in x, and $\delta \in \Delta$ with $\Delta = \{\delta \in \mathbb{R}^{p,m} : \|\delta\| \le 1\}$. Then the semi-infinite condition on x

$$\mathcal{F}(x) + \mathcal{L}\delta\mathcal{R}(x) + \mathcal{R}^T(x)\delta^T\mathcal{L}^T \preceq 0, \quad \forall\delta \in \Delta \tag{5}$$

is equivalent to the standard LMI condition on x and a slack variable τ

$$\begin{bmatrix} \mathcal{F}(x) + \tau\mathcal{L}\mathcal{L}^T & \mathcal{R}^T(x) \\ \mathcal{R}(x) & -\tau I \end{bmatrix} \preceq 0. \tag{6}$$

This lemma allows to replace the semi-infinite and possibly uncountable set of constraints in (5)—one for each possible value of δ in Δ—with the single LMI constraint (6) expressed over an enlarged set of variables. For additional details, the reader is referred to El Ghaoui et al. (1998). Extensions of this result exist, dealing with situations where the set Δ contains elements with block-diagonal structure, i.e., $\Delta = \{\delta = \mathrm{diag}(\delta_1, \ldots, \delta_\ell),\ \delta_i \in \mathbb{R}^{p_i,m_i} : \|\delta_i\| \le 1\}$. In this case, sufficient LMI conditions for the satisfaction of the semi-infinite constraint in (3) have been derived in El Ghaoui et al. (1998). In Section 4, we shall see an application of the above results to problems arising in truss structure design.

2.2 Average approach

In the average approach, one aims at solving the stochastic program

$$\min_{x\in\mathcal{X}} E_\delta[f(x,\delta)] \tag{7}$$

where $E_\delta[f(x,\delta)]$ is the expectation of $f(x,\delta)$ with respect to the probability measure $\mathbb{P}$ defined on $\delta \in \Delta$. Problem (7) is a classical stochastic optimization problem, and it has been extensively treated in the literature (see, for instance, Kushner and Yin 2003; Marti 2005; Ruszczyński and Shapiro 2003, and the references therein). Determining an exact solution for a stochastic program is, in general, computationally prohibitive (indeed, just evaluating the expectation for a fixed x amounts to computing a multi-dimensional integral, which might be numerically hard). There exist, however, simple situations where the expectation can be computed explicitly; hence, the problem converts to a standard convex program. For instance, a simple situation arises when $f(x,\delta)$ is a quadratic function in δ, i.e.,

$$f(x,\delta) = \delta^T A(x)\delta + b^T(x)\delta + c(x).$$

In this case, it is straightforward to see that the expectation can be explicitly computed if the first two moments of δ are given:

$$E_\delta[f(x,\delta)] = \mathrm{tr}\left(A(x)E_\delta[\delta\delta^T]\right) + b^T(x)E_\delta[\delta] + c(x).$$

3 Sampling-based approximate solutions

In this section, we propose a randomized approach for approximate solutions of both worst-case and average problems that cannot be solved efficiently via exact methods.

The main idea in randomized relaxations is a simple one: we collect a finite number N of random samples of the uncertainty

$$\delta^{(1)}, \delta^{(1)}, \ldots, \delta^{(N)}$$

extracted independently according to $\mathbb{P}$, and we construct a suitable "sampled" approximation of the problems previously considered.

We next show how to construct these approximations, and we highlight the theoretical properties of the solutions obtained via the approximated problems.

3.1 Worst-case design

A sampled approximation of problem (1) can be naturally stated as

$$\min_{x\in\mathcal{X}} \max_{i=1,\ldots,N} f(x,\delta^{(i)}).$$

This problem may also be rewritten in epigraph form as

$$\min_{x\in\mathcal{X}} t \quad \text{subject to}$$
$$f(x,\delta^{(i)}) \le t, \quad \text{for } i = 1,\ldots,N. \tag{8}$$

Notice that the possibly infinite number of constraints of problem (2) is substituted in problem (8) by a finite number N of *sampled scenarios* of the uncertainty. This scenario approximation of the worst-case problem has been first introduced in Calafiore and Campi (2005), where it has been shown that the solution obtained from the scenario problem is actually approximately feasible for the original worst-case problem, in a sense explained next.

For any given t, we define the *probability of violation* of x as

$$P_V(x,t) \doteq \mathbb{P}\{\delta \in \Delta : \; f(x,\delta) - t > 0\}.$$

For example, if a uniform (with respect to Lebesgue measure) probability distribution is assumed, then $P_V(x,t)$ measures the volume of 'bad' parameters δ such that the constraint $f(x,\delta) \leq t$ is violated. Clearly, a solution x with small associated $P_V(x,t)$ is feasible for most of the problem instances; i.e., it is *approximately feasible* for the robust problem.

Definition 1 (ϵ-level solution) Let $\epsilon \in (0,1)$. We say that $x \in \mathcal{X}$ is an ϵ-level robustly feasible (or, more simply, an ϵ-level) solution if $P_V(x,t) \leq \epsilon$.

The following theorem establishes the probabilistic properties of the scenario solution.

Theorem 1 (Corollary 1 of Calafiore and Campi 2006) *Assume that, for any extraction of* $\delta^{(1)}, \ldots, \delta^{(N)}$, *the scenario problem* (8) *attains a unique optimal solution* $\hat{x}_{\text{WC}}^{(N)}$. *Fix two real numbers* $\epsilon \in (0,1)$ *(robustness level) and* $\beta \in (0,1)$ *(confidence parameter) and let*[1]

$$N \geq N_{\text{WC}}(\epsilon,\beta) \doteq \left\lceil \frac{2}{\epsilon}\ln\frac{1}{\beta} + 2(n+1) + \frac{2(n+1)}{\epsilon}\ln\frac{2}{\epsilon} \right\rceil$$

then, with probability no smaller than $(1-\beta)$, $\hat{x}_{\text{WC}}^{(N)}$ *is* ϵ*-level robustly feasible.*

Note that this bound is a simplification of a tighter (but more involved) bound given in Theorem 1 of Calafiore and Campi (2006). We also remark that this result holds for any probability distribution $\mathbb{P}$ on δ; i.e., it is *distribution independent*. In many specific problem instances, the distribution of uncertainties is known (for instance, Gaussian); hence, the random samples $\delta^{(1)}, \ldots, \delta^{(N)}$ are extracted according to this distribution. If this is not the case, a uniform distribution on Δ might be assumed due to its worst-case properties (see, for instance, Barmish and Lagoa 1997); hence, $\delta^{(1)}, \ldots, \delta^{(N)}$ must be extracted uniformly.

[1]The notation $\lceil\,\rceil$ denotes the smallest integer greater than or equal to the argument.

3.2 Average design

For notation ease, we define the function

$$\phi_{\text{AV}}(x) \doteq E_\delta[f(x,\delta)].$$

We denote by x^*_{AV} a minimizer of $\phi_{\text{AV}}(x)$ and define the achievable minimum as

$$\phi^*_{\text{AV}} \doteq \min_{x\in\mathcal{X}} \phi_{\text{AV}}(x) = \phi_{\text{AV}}(x^*_{\text{AV}}).$$

We state a further assumption on the function f, namely, that the total variation of the function is bounded.

Assumption 2 *Let* $f^*(\delta) \doteq \min_{x\in\mathcal{X}} f(x,\delta)$, *and assume that the total variation of* f *is bounded by a constant* $V > 0$, i.e.,

$$f(x,\delta) - f^*(\delta) \leq V, \quad \forall x \in \mathcal{X}, \quad \forall \delta \in \Delta.$$

This implies that the total variation of the expected value is also bounded by V, i.e.,

$$\phi_{\text{AV}}(x) - \phi^*_{\text{AV}} \leq V, \quad \forall x \in \mathcal{X}.$$

Notice that we only assume that there exists a constant V such that the above holds, but we do not need to actually know its numerical value.

An approximate solution to (7) may be obtained by constructing an empirical estimate of ϕ_{AV} based on random samples:

$$\hat{\phi}^{(N)}_{\text{AV}}(x) \doteq \frac{1}{N}\sum_{i=1}^{N} f(x,\delta^{(i)}).$$

The function $\hat{\phi}^{(N)}_{\text{AV}}$ is convex in x, being the sum of convex functions.

Now, we study the convergence of the empirical minimum $\hat{\phi}^{(N)}_{\text{AV}}(\hat{x}_k)$ to the actual unknown minimum $\phi_{\text{AV}}(x^*)$. To this end, notice first that as x varies over $\mathcal{X}$, $f(x,\cdot)$ spans a family $\mathcal{F}$ of measurable functions of δ, namely, $\mathcal{F} \doteq \{f(x,\delta) : x \in \mathcal{X}\}$. A key step for assessing convergence is to bound (in probability) the relative deviation between the actual mean $\phi_{\text{AV}}(x) = E_\delta[f(x,\delta)]$ and the empirical mean $\hat{\phi}^{(N)}_{\text{AV}}(x)$ *for all* $f(\cdot,\delta)$ belonging to the family $\mathcal{F}$. In other words, for the given relative scale error $\epsilon \in (0,1)$, we require that

$$\text{PR}\left\{\sup_{x\in\mathcal{X}} \frac{|\phi_{\text{AV}}(x) - \hat{\phi}^{(N)}_{\text{AV}}(x)|}{V} > \epsilon\right\} \leq \alpha(N), \tag{9}$$

with $\alpha(N) \to 0$ as $N \to \infty$. Notice that the uniformity of the bound (9) with respect to x is crucial, as x is *not* fixed and known in advance: The uniform "closeness" of $\hat{\phi}^{(N)}_{\text{AV}}(x)$ to $\phi_{\text{AV}}(x)$ is the feature that allows us to perform the minimization on $\hat{\phi}^{(N)}_{\text{AV}}(x)$ instead of on

$\phi_{\text{AV}}(x)$. Property (9) is usually referred to as the uniform convergence of the empirical mean (UCEM) property. A fundamental result of the Learning Theory states that the UCEM property holds for a function class $\mathcal{F}$ whenever a particular measure of the complexity of the class, called the P-dimension of $\mathcal{F}$ (P-DIM ($\mathcal{F}$)), is finite. The interested reader can refer to the monographs (Vapnik 1998; Vidyasagar 1997) for formal definitions and further details.

The result for average optimization is given in the next theorem; see the Appendix for a proof.

Theorem 2 *Let* $\alpha, \epsilon \in (0, 1)$*, let* d *be an upper bound on the P-dim of* $\mathcal{F}$ *and let*

$$N \geq N_{\text{AV}} \doteq \frac{128}{\epsilon^2}\left[\ln\frac{8}{\alpha} + d\left(\ln\frac{32e}{\epsilon} + \ln\ln\frac{32e}{\epsilon}\right)\right].$$

Let x^*_{AV} *be a minimizer of* $\phi_{\text{AV}}(x)$*, and let* $\hat{x}^{(N)}_{\text{AV}}$ *be a minimizer of the empirical mean* $\hat{\phi}^{(N)}_{\text{AV}}(x)$*. Then, it holds with probability at least* $(1-\alpha)$ *that*

$$\frac{\phi_{\text{AV}}(\hat{x}^{(N)}_{\text{AV}}) - \phi_{\text{AV}}(x^*_{\text{AV}})}{V} \leq \epsilon.$$

That is, $\hat{x}^{(N)}_{\text{AV}}$ *is an* ϵ*-suboptimal solution (in the relative scale), with high probability* $(1-\alpha)$*. A solution* $\hat{x}^{(N)}_{\text{AV}}$ *such that the above holds is called an* $(1-\alpha)$*-probable* ϵ*-near minimizer of* $\phi_{\text{AV}}(x)$*, in the relative scale* V*.*

4 Applications to robust truss topology design

In this section, we apply some of the techniques discussed in the previous paragraphs to problems of truss topology optimization under uncertainty on the load pattern and/or on the material characteristics. See Ben-Tal and Nemirovski (1997) for other references dealing with the worst-case approach to truss topology design.

We consider a truss structure with m nodes connected by n bars in a given and fixed geometry. Nodes can be static (constrained) or free. The allowable movements of the nodes define the number M of degrees of freedom of the structure, with $M \leq m\nu$, being $\nu = 2$ for planar structures and $\nu = 3$ for three-dimensional structures. Let $\zeta \in \mathbb{R}^M$ denote the vector of components of external forces (loadings) applied on the free nodes of the structure, and let $d \in \mathbb{R}^M$ denote the vector of component node elastic displacements caused by the load ζ at equilibrium.

Denote with x_i the cross-sectional area of the ith bar, by ℓ_i the bar length, and by ϑ_i the Young's modulus of the bar. For small displacements, the following linearized relation holds

$$\zeta = K(x,\vartheta)d, \quad K(x,\vartheta) \doteq A\begin{bmatrix} \frac{\vartheta_1}{\ell_1}x_1 & & \\ & \ddots & \\ & & \frac{\vartheta_n}{\ell_n}x_n \end{bmatrix}A^T,$$

where $K(x,\vartheta)$ is the *stiffness matrix* of the structure and A is an $M \times n$ matrix that describes the geometry of the truss, that is, the kth column a_k of A is a vector of direction cosines such that $a_k^T d$ measures the elongation of the kth bar. The compliance of the truss is the work performed by the truss, which is given by

$$f(x,\zeta,\vartheta) = \frac{1}{2}\zeta^T d = \frac{1}{2}\zeta^T K^{-1}(x,\vartheta)\zeta. \tag{10}$$

The compliance, thus, represents the elastic energy stored in the structure. The optimization objective that we consider here is to minimize the compliance (10), hence maximizing the stiffness, subject to a constraint on the total volume of the structure

$$V(x) = \sum_{i=1}^{n} \ell_1 x_i = \ell^T x \leq V_{\max} \tag{11}$$

and to upper and lower bounds on the cross-sections:

$$x_{\text{lb}} \leq x \leq x_{\text{up}} \tag{12}$$

(vector inequalities are intended component-wise). To this end, notice first that minimizing $f(x,\zeta,\vartheta)$ is equivalent to minimizing a new slack variable γ under the additional constraint $f(x,\zeta,\vartheta) \leq \gamma$. Then, if the loading pattern ζ and the elastic coefficients ϑ_i, $i = 1,\ldots,n$ are known and fixed, the nonlinear optimization problem that we intend to solve is

$$\min_{\gamma,x} \gamma, \quad \text{subject to (11), (12),} \quad \text{and} \quad \zeta^T K^{-1}(x,\vartheta)\zeta \leq \gamma.$$

Notice further that the Schur complements rule (see, e.g., Section 2.1 of Boyd et al. 1994) permits to convert the nonlinear (and non-convex) constraint $f(x,\zeta,\vartheta) \leq \gamma$ into the convex LMI constraint

$$\begin{bmatrix} K(x,\vartheta) & \zeta \\ \zeta^T & \gamma \end{bmatrix} \succeq 0.$$

Therefore, for fixed loading pattern ζ and coefficients ϑ_i, $i = 1,\ldots,n$, the problem simply amounts to the solution of the following (convex) SDP in the design variable x and slack variable γ:

$$\begin{aligned}
\min_{\gamma,x} \gamma \quad & \text{subject to:} \\
& \ell^T x \le V_{\max} \\
& x_{\text{lb}} \le x \le x_{\text{up}} \\
& \begin{bmatrix} K(x,\vartheta) & \zeta \\ \zeta^T & \gamma \end{bmatrix} \succeq 0.
\end{aligned} \tag{13}$$

This problem may also be recast as a second-order cone program, which allows for additional efficiency in the numerical solution (see Ben-Tal and Bendsøe 1993; Ben-Tal and Nemirovski 1994; Lobo et al. 1998).

Remark 1 Notice that other types of truss design problems can be formulated in SDP form. For instance, within the previous setting, one can minimize the total volume $V(x)$ of the structure under a constraint $f(x,\zeta,\vartheta) \le \gamma$ on the compliance. This results in the following convex SDP problem:

$$\begin{aligned}
\min_{x} \ell^T x \quad & \text{subject to:} \\
& x_{\text{lb}} \le x \le x_{\text{up}} \\
& \begin{bmatrix} K(x,\vartheta) & \zeta \\ \zeta^T & 2\gamma \end{bmatrix} \succeq 0.
\end{aligned}$$

Alternatively, the fundamental modal frequency of the structure can be optimized, or lower bounds on this frequency can be imposed. If $M(x)$ (a symmetric, positive, semi-definite matrix affine in x) is the mass matrix of the structure, the fundamental modal frequency $\bar{\omega}$ is the square root of the smallest generalized eigenvalue of the pair (M,K), that is

$$\bar{\omega}^2 = \lambda_{\min}(M,K).$$

It is easy to show (see Vandenberghe and Boyd 1999) that, for a given $\Omega \ge 0$,

$$\bar{\omega} \ge \Omega \quad \Leftrightarrow \quad M(x)\Omega^2 - K(x) \preceq 0,$$

and this latter condition is an LMI in x. For instance, the problem of minimizing the total volume of the structure under the condition that $\bar{\omega} \ge \Omega$ is cast as the following SDP in x:

$$\begin{aligned}
\min_{x} V(x) \quad & \text{subject to:} \\
& x_{\text{lb}} \le x \le x_{\text{up}} \\
& M(x)\Omega^2 - K(x) \preceq 0.
\end{aligned}$$

On the other hand, there also exist important problems in structural optimization that may not be cast in the form of convex programs. A notable example is given, for instance, by stress-constrained problems in which bounds on the compression and tension on the individual bars are imposed together with local buckling constraints (Stolpe and Svanberg 2003). When buckling constraints are not present, the stress-constrained problem can be recast and solved by linear programming (see, for instance, Stolpe and Svanberg 2004). In this context, uncertainty in the loading pattern and material characteristic is usually dealt with using stochastic programming (see Marti 1999). For an overview of stress-constrained optimization, see Rozvany (2001) and references therein.

In the sequel, we consider a situation where the loading pattern and the material characteristics are uncertain in problem (13) and discuss robust design approaches.

4.1 Worst-case design

We first follow a robust approach and look for a design vector x that minimizes the worst-case (with respect to allowable loading patterns and Young's moduli) compliance of the structure. We consider three different cases, namely, polytopic uncertainty on both ϑ and ζ, norm-bounded uncertainty on ζ only, and a general case.

4.1.1 Polytopic uncertainty

Suppose that the loading vector ζ and the Young's moduli vector ϑ are only known to belong to a given polytope $\mathcal{P}$

$$(\zeta,\vartheta) \in \mathcal{P} \doteq \text{co}\{(\zeta,\vartheta)_1,\ldots,(\zeta,\vartheta)_v\}$$

where co{} denotes the convex hull and $(\zeta,\vartheta)_i$ denotes the ith vertex of $\mathcal{P}$. Notice that this setup encompasses, for instance, the case when the elements of ζ and/or ϑ belong to independent intervals. The following proposition holds; see the Appendix for a proof.

Proposition 1 *A solution to the robust optimization problem*

$$\begin{aligned}
\min_{x} \max_{(\zeta,\vartheta)\in\mathcal{P}} \zeta^T K^{-1}(x,\vartheta)\zeta \quad & \text{subject to:} \\
& \ell^T x \le V_{\max} \\
& x_{\text{lb}} \le x \le x_{\text{up}}
\end{aligned}$$

is obtained by solving the following convex SDP

$$\min_{x,\gamma} \gamma \quad \text{subject to:}$$

$$\ell^T x \leq V_{\max}$$

$$x_{\text{lb}} \leq x \leq x_{\text{up}}$$

$$\begin{bmatrix} K(x, \vartheta_i) & \zeta_i \\ \zeta_i^T & \gamma \end{bmatrix} \succeq 0, \quad i = 1, \ldots, v. \tag{14}$$

4.1.2 Norm-bounded uncertainty on loadings

Suppose ϑ is known while the loading pattern ζ is assumed to belong to an ellipsoid $\mathcal{E}$ of center ζ_c and shape matrix W, i.e.,

$$\mathcal{E} = \{\zeta = \zeta_c + Wz, \ \|z\| \leq 1\}. \tag{15}$$

For instance, in Ben-Tal and Nemirovski (1997), this ellipsoid is used to represent the envelope of the primary loadings and small "occasional loads." The following proposition holds; see the Appendix for a proof.

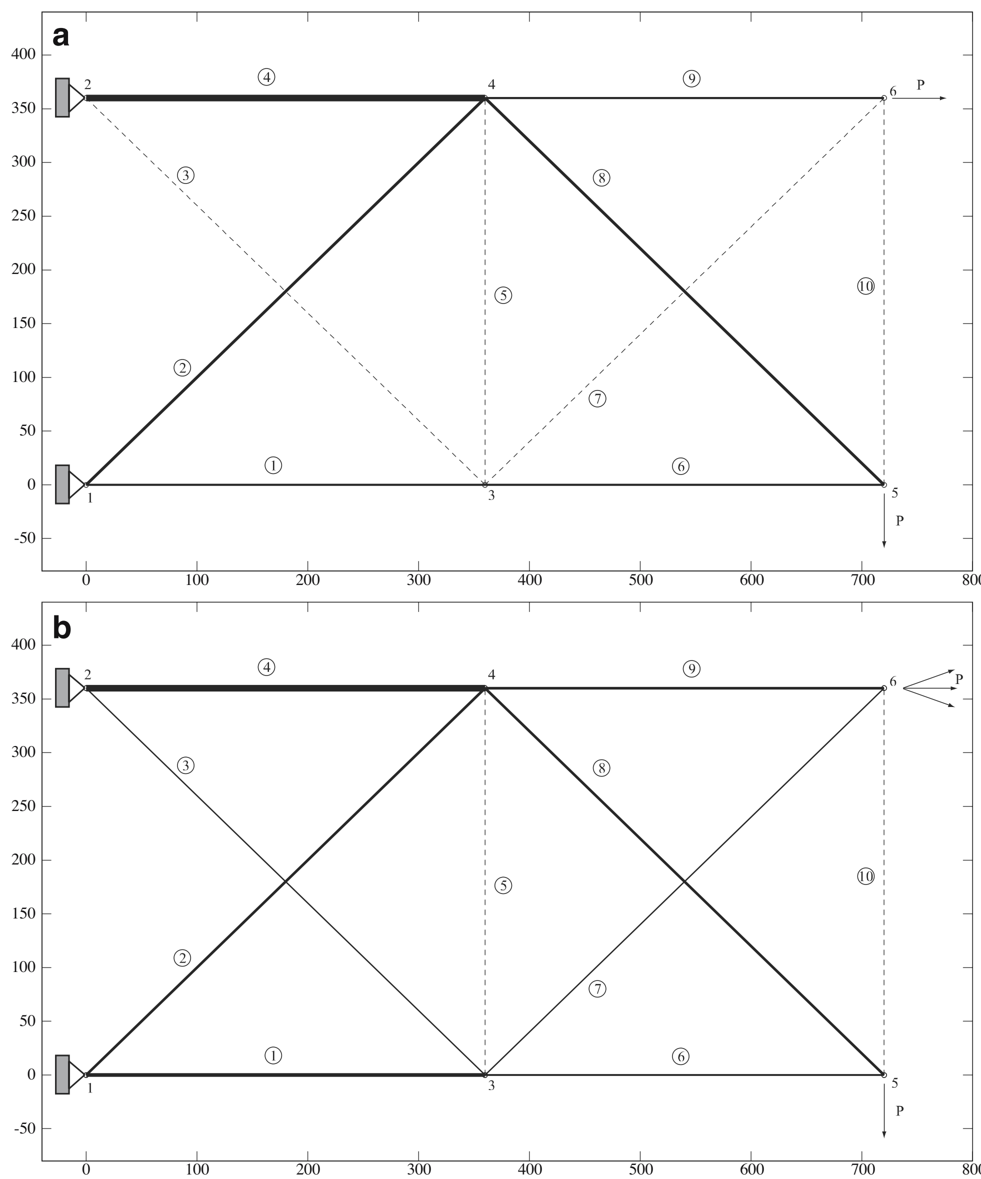

Fig. 1 **a** Design resulting from nominal load. **b** Robust design under interval uncertainty on the loading of node 6

Proposition 2 *A solution to the robust optimization problem*

$$\min_x \max_{\zeta\in\mathcal{E}} \zeta^T K^{-1}(x,\vartheta)\zeta \quad \text{subject to:} \quad \ell^T x \le V_{\max}, \quad x_{\text{lb}} \le x \le x_{\text{up}} \tag{16}$$

is obtained by solving the following convex SDP

$$\min_{x,\gamma,\tau} \gamma \quad \text{subject to:}$$
$$\ell^T x \le V_{\max}$$
$$x_{\text{lb}} \le x \le x_{\text{up}}$$
$$\begin{bmatrix} K(x,\vartheta) - \tau W W^T & \zeta_{\text{c}} & 0_{M,1} \\ {\zeta_{\text{c}}}^T & \gamma & 1 \\ 0_{1,M} & 1 & \tau \end{bmatrix} \succeq 0.$$

4.1.3 Generic uncertainty on loadings and Young's moduli

In the case when both ζ and ϑ vary over generic sets, it is not possible in general to obtain *exact* reformulation of the worst-case design problem in terms of SDPs. However, the problem can still be posed in a probabilistically approximate sense by following the sampling approach described in Section 3.1. Specifically, let $\zeta \in Z$ and $\vartheta \in \Theta$, and assume that these uncertainties are of random nature and that samples $\zeta^{(i)} \in Z$, $\vartheta^{(i)} \in \Theta$, and $i = 1, \ldots, N$ can be generated according to the underlying probability distribution $\mathbb{P}$ over $Z \times \Theta$. Once the probabilistic levels ϵ, β are fixed, Theorem 1 provides a bound on the required number of samples, and an approximate solution to the robust design problem can be obtained by solving the following SDP:

$$\min_{x,\gamma} \gamma \quad \text{subject to:}$$
$$\ell^T x \le V_{\max}$$
$$x_{\text{lb}} \le x \le x_{\text{up}}$$
$$\begin{bmatrix} K(x,\vartheta^{(i)}) & \zeta^{(i)} \\ \zeta^{(i)T} & \gamma \end{bmatrix} \succeq 0, \quad i = 1, \ldots, N. \tag{17}$$

4.2 Average design

In an average design approach, we look for a solution vector x that minimizes the expected value of the compliance. We assume that ϑ (the Young's moduli) is exactly known, whereas ζ (the loading) is random with known expected value $\bar{\zeta} \doteq E\{\zeta\}$ and covariance matrix

$$\Gamma \doteq E\{(\zeta - \bar{\zeta})(\zeta - \bar{\zeta})^T\} = E\{\zeta\zeta^T\} - \bar{\zeta}\bar{\zeta}^T.$$

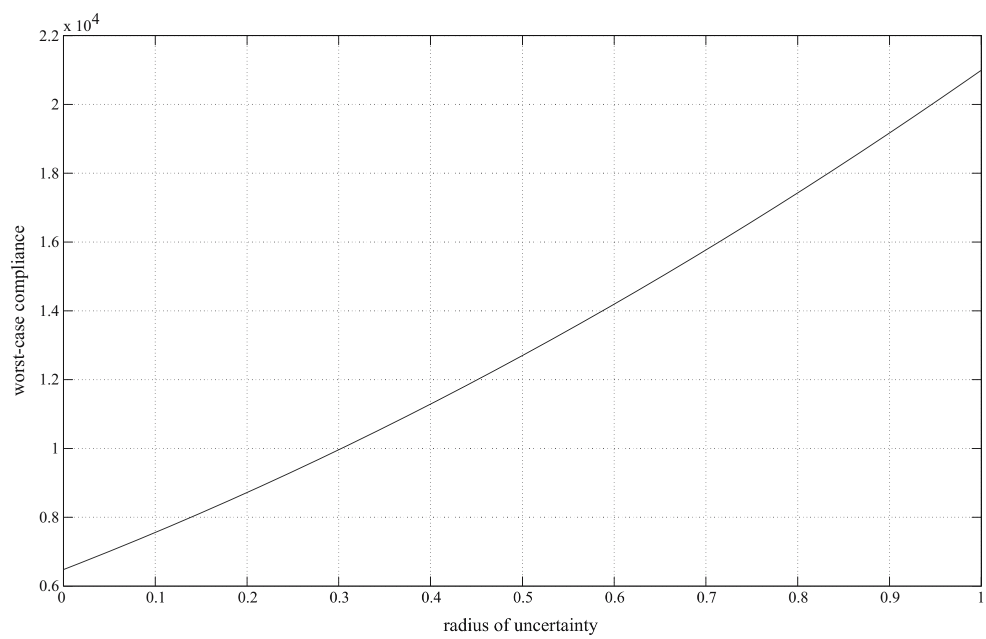

Fig. 2 Worst-case compliance as a function of uncertainty radius r

In this case, we have explicitly

$$E_\zeta\{\zeta^T K^{-1}(x,\vartheta)\zeta\} = \mathrm{tr}\, E_\zeta\{K^{-1}(x,\vartheta)\zeta\zeta^T\} = \mathrm{tr}\left(K^{-1}(x,\vartheta)(\Gamma + \bar{\zeta}\bar{\zeta}^T)\right) = \bar{\zeta}^T K^{-1}(x,\vartheta)\bar{\zeta} + \mathrm{tr}\left(K^{-1}(x,\vartheta)\Gamma\right).$$

Introducing two slack variables γ and $X = X^T$, the problem

$$\min_x E_\zeta\{\zeta^T K^{-1}(x,\vartheta)\zeta\} \quad \text{subject to:} \quad \ell^T x \le V_{\max} \quad x_{\mathrm{lb}} \le x \le x_{\mathrm{up}} \tag{18}$$

can be equivalently rewritten as the following convex SDP problem (see Vandenberghe and Boyd 1999, for details on the derivation):

$$\min_{x,\gamma,X} \gamma + \mathrm{tr}\,\Gamma X \quad \text{subject to:} \quad \ell^T x \le V_{\max} \quad x_{\mathrm{lb}} \le x \le x_{\mathrm{up}} \quad \begin{bmatrix} K(x,\vartheta) & \bar{\zeta} \\ \bar{\zeta} & \gamma \end{bmatrix} \succeq 0 \quad \begin{bmatrix} K(x,\vartheta) & I \\ I & X \end{bmatrix} \succeq 0.$$

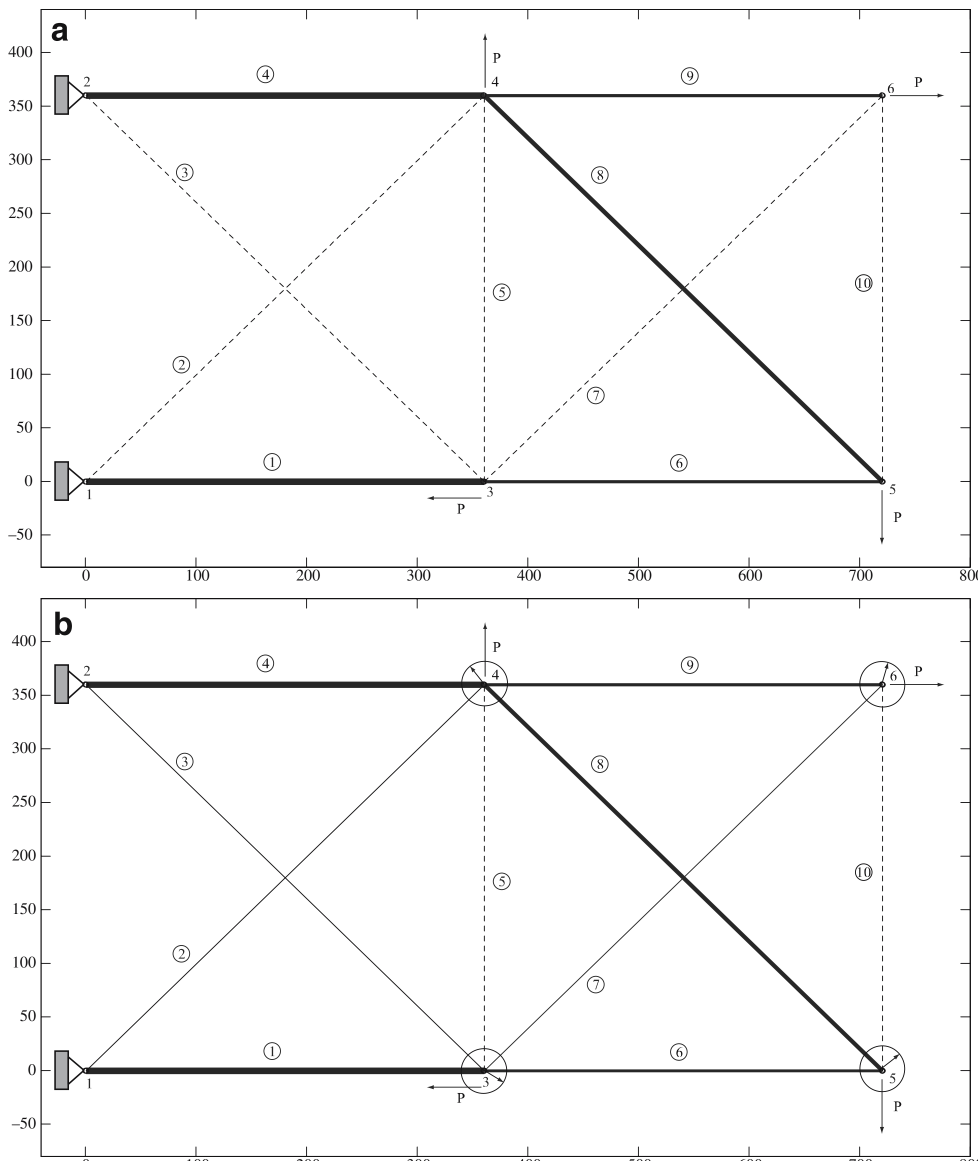

Fig. 3 **a** Optimal design for nominal load. **b** Probabilistically robust design resulting from the scenario solution under spherical uncertainty on loadings of nodes 3, 4, 5, and 6

4.2.1 Approximate solution for generic uncertainty

In a general situation of joint random uncertainty on ζ and ϑ, we may resort to sampling approximation of the expectation, as discussed in Section 3.2. In practice, the objective in (18) is replaced by the sample mean, and it can be shown that the resulting approximate problem can be cast in SDP form as follows:

$$\min_{x,t} \frac{1}{N}\sum_{i=1}^{N} t_i \quad \text{subject to:}$$
$$\ell^T x \le V_{\max}$$
$$x_{\text{lb}} \le x \le x_{\text{up}}$$
$$\begin{bmatrix} K(x,\vartheta^{(i)}) & \zeta^{(i)} \\ \zeta^{(i)T} & t_i \end{bmatrix} \succeq 0, \quad i = 1,\dots,N,$$

where $\vartheta^{(i)}$ and $\zeta^{(i)}$ are the randomly extracted samples of the uncertainties.

5 Numerical examples

In this section, we present some simple numerical examples of robust truss optimization. We first consider a planar ten-bar example from Marti (1999), assuming $x_{\text{lb}} = 0$, $x_{\text{ub}} = 100$ mm^2, $V_{\max} = 10^5$ mm^3, Young's moduli $\theta_i = 10^4$ N/mm^2, and loading intensity $P = 1,000$ N.

The solution of problem (13) under nominal load pattern is depicted in Fig. 1a.

The optimal compliance resulted to be $\gamma_{\text{nom}} = 6,480$ N mm, and the optimal bar sections

$$x_{\text{nom}} = \left[\, 27.78\ 39.28\ 0\ 83.33\ 0\ 27.78\ 0\ 39.28\ 27.78\ 0 \,\right]^T.$$

Next, we "robustified" this design by considering a set of loading patterns where the loading on node 6 is $(P, \delta P)$, with $\delta \in [-r, r]$. This situation falls in the polytopic setting discussed in Section 4.1.1. Solving the SDP (14) with uncertainty level $r = 0.5$, we obtained the solution depicted in Fig. 1b. The worst-case optimal compliance resulted to be $\gamma_{\text{wc}} = 12,701$ N mm, and the optimal bar sections

$$x_{\text{wc}} = \left[\, 39.68\ 28.06\ 14.03\ 69.44\ 0\ 19.84\ 14.03\ 28.06\ 29.76\ 0 \,\right]^T.$$

Figure 2 shows the increase in the worst-case compliance as the uncertainty radius r varies from 0 (nominal load pattern) to 1.

Finally, we assume a nominal loading pattern as the one considered in Ben-Tal and Nemirovski (1997). The optimal solution of problem (13) under this nominal load is depicted in Fig. 3a. The optimal compliance resulted to be $\gamma_{\text{nom}} = 4,147$ N mm, and the optimal bar sections

$$x_{\text{nom}} = \left[\, 69.44\ 0\ 0\ 69.44\ 0\ 34.72\ 0\ 49.10\ 34.72\ 0 \,\right]^T.$$

On this configuration, we next considered the following uncertainty on the loadings: In addition to the nominal loading, each node is subject to a force having random direction and intensity 50 N. This worst-case design problem was solved using the sampling technique discussed in Section 4.1.3. Specifically, setting robustness level $\epsilon = 0.05$ and confidence $\beta = 10^{-10}$, the best bound in Calafiore and Campi (2006) requires $N = 1,651$ samples. Problem (17) then yielded the probabilistically robust solution depicted in Fig. 3b, having probable worst-case compliance $\hat{\gamma}^{(N)}{}_{\text{wc}} = 9,527$ and bar sections

$$\hat{x}_{\text{wc}}{}^{(N)} = \left[\, 70.16\ 0.05\ 4.36\ 66.68\ 0\ 32.42\ 2.26\ 45.83\ 34.23\ 0 \,\right]^T.$$

6 Conclusions

In this paper, we discussed worst-case and average approaches for convex programming under uncertainty. Optimal solutions can be determined efficiently in some special situations, whereas the general case can be tackled by means of sampling-based probabilistic relaxations. These methods have the advantage of being conceptually simple and easy to apply, thus representing an appealing design option also for the practitioner. As shown in Section 4, robust and sampling-based optimization methods can be applied with success to design problems arising in structural mechanics. The methods presented here are all based on batch solutions to SDP problems with many constraints. This may represent a problem in practice for large-scale applications. Alternative *iterative* (i.e., sequential, non-batch) techniques have been developed for feasibility problems (see Calafiore and Dabbene 2007), while extension to optimization problems is the subject of current research.

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Appendix

Proof of Lemma 1 The implication from left to right in (4) is obvious; we, hence, consider only the reverse implication. Let x be such that the right-hand side of (4) holds, and let δ be any point in the polytope Δ.

Then, there exist $\alpha_1, \ldots, \alpha_v \geq 0$, $\sum_{j=1}^{v} \alpha_j = 1$ such that $\delta = \sum_{j=1}^{v} \alpha_j \delta_j$. Since $F(x, \delta)$ is affine in δ, we may write it as

$$F(x, \delta) = F(x, 0) + \tilde{F}(x, \delta),$$

where $\tilde{F}(x, \delta)$ is *linear* in δ. Therefore,

$$\begin{aligned} F(x, \delta) &= F\left(x, \sum_{j=1}^{v} \alpha_j \delta_j\right) \\ &= F(x, 0) + \tilde{F}\left(x, \sum_{j=1}^{v} \alpha_j \delta_j\right) \\ &= F(x, 0) + \sum_{j=1}^{v} \alpha_j \tilde{F}(x, \delta_j) \\ &= \sum_{j=1}^{v} \alpha_j F(x, 0) + \sum_{j=1}^{v} \alpha_j \tilde{F}(x, \delta_j) \\ &= \sum_{j=1}^{v} \alpha_j F(x, \delta_j) \preceq 0, \end{aligned}$$

which proves the statement. □

Proof of Theorem 2 Consider the function family $\mathcal{G}$ generated by the functions

$$g(x, \delta) \doteq \frac{f(x, \delta) - f^*(\delta)}{V},$$

as x varies over $\mathcal{X}$. The family $\mathcal{G}$ is a simple rescaling of $\mathcal{F}$ and maps Δ into the interval [0, 1]; therefore, the P-dimension of $\mathcal{G}$ is the same as that of $\mathcal{F}$. Define

$$\phi_g(x) \doteq E_\delta[g(x, \delta)] = \frac{\phi_{\text{AV}}(x) - K}{V}, \tag{19}$$

and

$$\hat{\phi}_g^{(N)}(x) \doteq \frac{1}{N} \sum_{i=1}^{N} g(x, \delta^{(i)}) = \frac{\hat{\phi}_{\text{AV}}^{(N)}(x) - \hat{K}}{V},$$

where

$$K \doteq E_\delta[f^*(\delta)], \quad \hat{K} \doteq [f^*(\delta)] = \frac{1}{N} \sum_{i=1}^{N} f^*(\delta^{(i)}).$$

Notice that a minimizer $\hat{x}_{\text{AV}}$ of $\hat{\phi}_{\text{AV}}^{(N)}(x)$ is also a minimizer of $\hat{\phi}_g^{(N)}(x)$. Then, Theorem 2 in Vidyasagar (2001) guarantees that, for $\alpha, \nu \in (0, 1)$,

$$\text{Pr}\left\{ \sup_{g \in \mathcal{G}} \left| E_\delta[g(\delta)] - \frac{1}{N} \sum_{i=1}^{N} g(\delta^{(i)}) \right| > \nu \right\} \leq \alpha,$$

provided that

$$N \geq \frac{32}{\nu^2} \left[\ln \frac{8}{\alpha} + \text{P-DIM}(\mathcal{G}) \left(\ln \frac{16e}{\nu} + \ln \ln \frac{16e}{\nu} \right) \right].$$

Applying this theorem with $\nu = \epsilon/2$ and using the bound P-DIM $(\mathcal{G})$ = P-DIM $(\mathcal{F}) \leq d$, we have that, for all $x \in \mathcal{X}$, it holds with probability at least $(1 - \alpha)$ that

$$|\phi_g(x) - \hat{\phi}_g^{(N)}(x)| \leq \frac{\epsilon}{2}. \tag{20}$$

From (20), evaluated in $x = x_{\text{AV}}^*$, it follows that

$$\phi_g(x_{\text{AV}}^*) \geq \hat{\phi}_g^{(N)}(x_{\text{AV}}^*) - \frac{\epsilon}{2} \geq \hat{\phi}_g^{(N)}(\hat{x}_{\text{AV}}^{(N)}) - \frac{\epsilon}{2}, \tag{21}$$

where the last inequality follows because $\hat{x}_{\text{AV}}^{(N)}$ is a minimizer of $\hat{\phi}_g$. From (20), evaluated in $x = \hat{x}_{\text{AV}}^{(N)}$, it follows that

$$\hat{\phi}_g^{(N)}(\hat{x}_{\text{AV}}^{(N)}) \geq \phi_g(\hat{x}_{\text{AV}}^{(N)}) - \frac{\epsilon}{2},$$

which substituted in (21) gives

$$\phi_g(\hat{x}_{\text{AV}}^{(N)}) \geq \phi_g(\hat{x}_{\text{AV}}^{(N)}) - \epsilon.$$

From the last inequality and (19), it follows that

$$\phi(\hat{x}_{\text{AV}}^{(N)}) - \phi(x_{\text{AV}}^*) \leq \epsilon V,$$

which concludes the proof. □

Proof of Proposition 1 First, notice that minimizing $\max_{(\zeta,\vartheta)\in\mathcal{P}} \zeta^T K^{-1}(x, \vartheta)\zeta$ is equivalent to minimizing a slack variable γ, subject to the additional constraint $\max_{(\zeta,\vartheta)\in\mathcal{P}} \zeta^T K^{-1}(x, \vartheta)\zeta \leq \gamma$. In turn, this latter constraint is equivalent to imposing

$$\zeta^T K^{-1}(x, \vartheta)\zeta \leq \gamma, \quad \forall (\zeta, \vartheta) \in \mathcal{P},$$

which is rewritten in robust SDP form using Schur complements:

$$\begin{bmatrix} K(x, \vartheta) & \zeta \\ \zeta^T & \gamma \end{bmatrix} \succeq 0, \quad \forall (\zeta, \vartheta) \in \mathcal{P}.$$

A direct application of Lemma 1 then easily concludes the proof. □

Proof of Proposition 2 Following the same steps as in the proof of Proposition 1, we have that problem (16) is equivalent to

$$\begin{aligned} \min_{x,\gamma,\tau} \gamma \quad & \text{subject to:} \\ & \ell^T x \leq V_{\max} \\ & x_{\text{lb}} \leq x \leq x_{\text{up}} \\ & \begin{bmatrix} K(x, \vartheta) & \zeta \\ \zeta^T & \gamma \end{bmatrix} \succeq 0, \quad \forall \zeta \in \mathcal{E}. \end{aligned}$$

Substituting (15), the latter constraint becomes

$$\begin{bmatrix} K(x,\vartheta) & \zeta_c \\ \zeta_c^T & \gamma \end{bmatrix} + \begin{bmatrix} W \\ 0_{M,1} \end{bmatrix} z \begin{bmatrix} 0_{1,M} & 1 \end{bmatrix} + (\text{last term})^T \succeq 0,$$

$\forall \zeta : \|z\| \leq 1.$

The statement now follows from direct application of Lemma 2 to the latter expression. □

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