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Automatica 39 (2003) 1773–1781

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Brief Paper

# Recursive algorithms for inner ellipsoidal approximation of convex polytopes☆

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Received 13 March 2002; received in revised form 28 October 2002; accepted 6 May 2003

**Abstract**

In this paper, fast recursive algorithms for the approximation of an $n$-dimensional convex polytope by means of an inscribed ellipsoid are presented. These algorithms consider at each step a single inequality describing the polytope and, under mild assumptions, they are guaranteed to converge in a finite number of steps. For their recursive nature, the proposed algorithms are better suited to treat a quite large number of constraints than standard off-line solutions, and have their natural application to problems where the set of constraints is iteratively updated, as on-line estimation problems, nonlinear convex optimization procedures and set membership identification.


*Keywords:* Estimation; Ellipsoidal approximation; Convex polytope

## 1. Introduction

The problem of approximating a convex polytope in $n$-dimensions by means of simpler geometrical shapes arises in many research fields related to optimization, system identification and control. The main reason for the need of such approximation is the fact that, as the number of inequalities describing the polytope grows, the amount of data to be processed and the consequent computational effort for its manipulation become critically large. For this reason, different kinds of approximations based on low complexity sets, depending on the particular application of interest, have been proposed in recent years.

Inner ellipsoidal approximation is a basic tool of efficient algorithms for solving nonlinear programming problems with convex objective and constraints, see Tarasov, Khachiyan, and Erlikh (1988), Khachiyan and Todd (1990) and Nesterov and Nemirovskii (1994), whereas in tolerance design ellipsoidal techniques based on successive inner approximation have been employed in the solution of design centering problems, see e.g. Wojciechowski and Vlach (1993). In both cases, the main step of the proposed algorithms consists in iteratively approximating the constraint region or the design specification. Similar problems arise also in set membership identification when considering linear in the parameters models, see e.g. Durieu, Walter, and Polyak (2001).

Motivated by the above considerations, in this paper we present efficient iterative algorithms for constructing inner ellipsoidal approximations of a convex polytope. In the past years, some different approaches have been proposed in the literature (see for instance Tarasov et al., 1988; Khachiyan & Todd, 1990; Nesterov & Nemirovskii, 1994; Vandenberghe & Boyd, 1999) for tackling this problem. Most of them are based on an LMI formulation. Unfortunately, when the number of inequalities to be processed increases, the computational burden related to this kind of off-line solutions easily becomes unaffordable with the standard software packages. Hence, the interest on iterative algorithms that consider at each step only one inequality describing the polytope.

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Kenko Uchida under the direction of Editor Tamer Basar.

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doi:10.1016/S0005-1098(03)00180-8

The paper is organized as follows. First, definitions and problem statement are introduced: the convex polytope is described by a finite number $N$ of inequalities, while the ellipsoid is expressed by its center $c$ and the positive definite shape matrix $P$. Then, in Section 3, a *feasibility problem* is addressed, i.e. for given $\gamma > 0$ we derive an iterative algorithm that finds, if it exists, an ellipsoid of size greater than $\gamma$ contained in the polytope. The algorithm is based upon subgradient iterations. At each step, a single bounding inequality is considered; if that constraint is violated by the current solution, a correction step towards the direction of the subgradient of a specially constructed scalar function is performed. The basic ideas of this method can be traced back to the method for solving linear inequalities with vector variables proposed by Yakubovich in 1960, see Bondarko and Yakubovich (1992). These techniques are here suitably extended (we consider various rules for selecting the order in which the constraints are processed) and improved to deal with inequalities of matrix variables, continuing the line of research of Calafiore and Polyak (2001) and Polyak and Tempo (2001). In Section 4, the optimization problem of finding the largest possible inscribed ellipsoid is considered. Clearly, an approximation of the largest inscribed ellipsoid can be obtained iterating the feasibility algorithm with increasing values of $\gamma$. However, an alternative method, that directly solves the optimization problem, is also presented. In this algorithm, the two steps of recursively considering the constraints and enlarging the ellipsoid are embedded. In Section 5, we compare the proposed algorithms with the known ones from computational point of view and discuss the situations when the proposed algorithms are more efficient. In Section 6, the algorithms are specialized to deal with set membership identification problems. Two application examples are presented in Section 7. Finally, in Section 8 some directions for future applications are addressed.

## 2. Preliminaries and problem statement

We denote by $\mathbb{S}^n$ the set of symmetric matrices $\mathbb{S}^n \doteq \{P \in \mathbb{R}^{n,n}\colon P = P^{\mathrm{T}}\}$. Given a matrix $P \in \mathbb{S}^n$, we use the notation $P \succ 0$ (resp. $P \succcurlyeq 0$) to indicate that the matrix $P$ is symmetric and positive (resp. nonnegative) definite. The convex cone of symmetric nonnegative definite matrices is denoted by $\mathscr{C} \doteq \{P \in \mathbb{S}^n : P \succcurlyeq 0\}$. For any symmetric matrix $P \in \mathbb{S}^n$ and any closed convex set $\mathscr{X} \subset \mathbb{R}^{n,n}$, we define the projection of $P$ on $\mathscr{X}$ as

$$\Pi_{\mathscr{X}}[P] \doteq \arg\min_{X \in \mathscr{X}} \|P - X\|, \tag{1}$$

where $\|\cdot\|$ is the standard Frobenius norm. The projection of $P$ on $\mathscr{C}$ can be easily computed as

$$\Pi_{\mathscr{C}}[P] = U_p E_p U_p^{\mathrm{T}}.$$

The matrices $U_p$ and $E_p$ are taken from the eigenvalue decomposition of $P$,

$$P = [U_p \;\; U_n] \begin{bmatrix} E_p & 0 \\ 0 & E_n \end{bmatrix} \begin{bmatrix} U_p^{\mathrm{T}} \\ U_n^{\mathrm{T}} \end{bmatrix},$$

where $E_p$ and $E_n$ are diagonal matrices containing nonnegative and strictly negative eigenvalues of $P$ respectively, and $[U_p \; U_n]$ is an orthogonal matrix.

We now formally state the problem addressed in the paper. Consider a convex polytope $\mathscr{D}$ described by a set of $N$ linear inequalities

$$\mathscr{D} = \{\theta \in \mathbb{R}^n\colon a_k^{\mathrm{T}}\theta \leqslant b_k, k = 1, \ldots, N\}, \tag{2}$$

that we assume to be nonempty, with nonempty interior and bounded. In order to approximate the above set, we consider ellipsoids of the form

$$\mathscr{E} \doteq \{\theta \in \mathbb{R}^n\colon \theta = Pz + c, \|z\| \leqslant 1\}, \tag{3}$$

where $c \in \mathbb{R}^n$, $P \succcurlyeq 0$ represent the ellipsoid center and the *shape* matrix, respectively, while $\|.\|$ is Euclidean norm. For ease of notation, these information are collected in a single parameter

$$x \doteq \begin{pmatrix} c \\ P \end{pmatrix}. \tag{4}$$

The ellipsoid $\mathscr{E}$ belongs to the set $\mathscr{D}$ if and only if it is contained in each one of the half-spaces that define (2). This fact is stated in the following lemma, which is proven, in slightly different forms, in Boyd, Ghaoui, Feron, and Balakrishnan (1994), Tarasov et al. (1988) and Vandenberghe and Boyd (1999).

**Lemma 1.** *The ellipsoid $\mathscr{E}$ defined in* (3) *is inscribed in the convex polytope $\mathscr{D}$ given in* (2) *if and only if the following condition holds*:

$$m_k(x) \doteq a_k^{\mathrm{T}} c + \|Pa_k\| - b_k \leqslant 0, \quad k = 1, \ldots, N. \tag{5}$$

We refer to the function $m_k(x)$ as *membership function*. For given $k$, the membership function is a scalar valued function of the matrix variables forming $x$ and can be viewed as a measure of the violation of the $k$th constraint. The introduction of the function $m_k(x)$ allows us to formulate the problem of finding the largest ellipsoid inscribed in $\mathscr{D}$ as the following optimization problem.

**Problem 1** (Largest inscribed ellipsoid).

$$\begin{aligned} &\text{maximize} && f(x) = \operatorname{Tr} P \\ &\text{subject to} && P \in \mathscr{C}, \\ &&& m_k(x) \leqslant 0, \quad k = 1, \ldots, N. \end{aligned} \tag{6}$$

The function $f(x)$ corresponds to the sum of the ellipsoid semiaxes. We remark that the choice of measuring the size of the ellipsoid by its volume would lead to the cost function $f(x) = \log\det P$. The function $\mathrm{Tr}\, P$ has some additional advantages, because it is a linear function of $P$.

The convexity of $m_k(x)$, together with the equations for the computation of the subgradients, is stated in the following lemma.

**Lemma 2.** *Optimization problem* (6) *is convex and admits an unique solution* $P \succ 0$. *Moreover, for fixed* $k$, *the scalar function* $m_k(x)$ *is convex in* $x$ *and its subgradient can be computed as*

$$\partial_x m_k(x) = \begin{pmatrix} \partial_c m_k(x) \\ \partial_P m_k(x) \end{pmatrix} = \begin{pmatrix} a_k \\ \frac{a_k a_k^{\mathrm{T}} P + P a_k a_k^{\mathrm{T}}}{2\|P a_k\|} \end{pmatrix},$$

*while* $\partial_P m_k(x) = 0$ *if* $P a_k = 0$

The proof of this lemma is omitted for brevity. The uniqueness of the solution and its positive definiteness may be proved using arguments similar to the ones in the proof of Theorem 3.1 in Durieu et al. (2001). The condition for the ellipsoid $\mathscr{E}$ to belong to the convex set $\mathscr{D}$ is therefore expressed as a set of $N$ scalar convex inequalities with matrix variables.

## 3. Feasibility problem

In order to find an iterative solution of the above problem, we first concentrate on a feasibility problem. That is, we fix some level $\gamma > 0$ for the cost function and consider the general problem of iteratively finding a feasible solution to the set of scalar inequalities

$$m_k(x) \leqslant 0, \quad k = 1, \ldots, N, \quad P \in \mathscr{Q}\gamma, \tag{7}$$

where $\mathscr{Q}\gamma$ is the set of nonnegative definite matrices with trace greater or equal than $\gamma$. For any matrix $P \in \mathbb{S}^n$, we introduce the matrix operator $\Pi_{\mathscr{Q}\gamma}[P]$, *a quasi-projection* on the set $\mathscr{Q}\gamma$, as

$$\Pi_{\mathscr{Q}\gamma}[P] = P_+ + \frac{(\gamma - \mathrm{Tr}\, P_+)_+}{n} I, \tag{8}$$

where $P_+ = \Pi_{\mathscr{C}}[P], (x)_+ = \max\{x, 0\}$. This operator is not a standard projection, in the sense that it does not satisfy (1). However, it shares some nice properties of projections, that we summarize in the following lemma, whose easy proof is omitted.

**Lemma 3.** *Given a matrix* $P \in \mathbb{S}^n$ *and* $\gamma > 0$, *its quasi-projection defined in* (8) *has the following properties*:

1. $\Pi_{\mathscr{Q}\gamma}[P] \in \mathscr{Q}\gamma \quad (9)$

2. $\|\Pi_{\mathscr{Q}\gamma}[P] - X\| \leqslant \|P - X\|, \quad \forall X \in \mathscr{Q}\gamma. \quad (10)$

First, we have to make sure that the problem we are dealing with is a feasible one, i.e. we need to assume that there exist at least one ellipsoid with trace greater than $\gamma$ contained in $\mathscr{D}$. Formally, the following strong feasibility assumption is supposed to hold

**Assumption** (Strong feasibility). *The solution set*

$$\mathscr{A}_\gamma \doteq \left\{ x = \begin{pmatrix} c \\ P \end{pmatrix} : c \in \mathbb{R}^n, P \in \mathscr{Q}\gamma, \right.$$
$$\left. m_k(x) \leqslant 0,\ k = 1, \ldots, N \right\}$$

*has a nonempty interior.*

This is equivalent to the assumption $\gamma < f_*$, where $f_*$ is the optimal value in (1).

As mentioned before, the proposed algorithm considers, at each step $i$, a single condition $m_{k_i}(x)$ expressed by inequality (5). Note that the notation $k_i$ is used to label the index corresponding to the constraint $a_{k_i}^{\mathrm{T}} \theta \leqslant b_{k_i}$ considered at the $i$th step. Various rules for the choice of the index $k_i$ can be considered, each one leading to different properties of the algorithm. In particular, we concentrate on the following ones.

**Rule 1** (Cyclical order). The constraints are considered cyclically, i.e. at the $i$th step the index $k_i$ is chosen as $k_i = i \bmod N$.

**Rule 2** (Most violated constraint). At the $i$th step, the index $k_i$ is chosen such that
$k_i = \arg\max_{k=1,\ldots,N} m_k(x_i)$.

**Rule 3** (Random order). The index $k_i$ is chosen at random. At each step the probability of choosing a given inequality is strictly greater than zero, i.e.
$p_i(k) \doteq \Pr\{k\text{th inequality is chosen at step } i\} = p(k) > 0,\ k = 1, \ldots, N$.

With the above rules defined, we first provide an intuitive geometrical interpretation of the proposed feasibility algorithm: At the $i$th step of the algorithm, if the constraint expressed by the corresponding linear inequality (chosen according to one of the above rules) is violated by the current ellipsoid, then the ellipsoid is projected into the feasibility half-space and finally its size is enlarged to the value of $\gamma$. If, on the contrary, the current ellipsoid does not violate the inequality, no correction step is needed and a new constraint is processed. Formally, the proposed iterative algorithm can be stated as follows.

**Algorithm 1.** *Consider a starting point* $x_0 = \binom{c_0}{P_0}$, $c_0 \in \mathbb{R}^n$, $P_0 \in \mathscr{Q}\gamma$ *and construct the following recursion*:

- (correction step) *if* $m_{k_i}(x_i) > 0$,

$$x_{i+1} = \begin{pmatrix} c_{i+1} \\ P_{i+1} \end{pmatrix} = \begin{pmatrix} c_i - \lambda_i \dfrac{\partial_c m_{k_i}(x_i))}{\|\partial_x m_{k_i}(x_i)\|} \\ \Pi_{\mathscr{Q}\gamma}[P_i - \lambda_i \dfrac{\partial_P m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|}] \end{pmatrix}, \quad (11)$$

*where the stepsize* $\lambda_i$ *is given by*

$$\lambda_i = \eta \left( \frac{m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|} + \alpha \right) \quad (12)$$

*being* $\eta \in (0,2)$ *a scalar parameter and* $0 < \alpha \leqslant r$, *where* $r$ *is the radius of a ball* $\mathscr{B}$ *contained in* $\mathscr{A}_\gamma$.
- (no correction step) *otherwise* $x_{i+1} = x_i$.

*The formulas for the subgradients in* (11) *and* (12) *are given in Lemma* 2, *and lead to* $\|\partial_x m_{k_i}(x_i)\|^2 = 2\|a_{k_i}\|^2$.

We remark that this algorithm has some common features with the algorithm presented by Calafiore and Polyak (2001) for the solution of robust feasibility problems under LMI constraint. A similar algorithm has also been applied in Polyak and Tempo (2001) for the design of probabilistic LQ regulators. However, the algorithms in these references are of random nature and deal with infinite number of LMI constraints. We are now ready to state the main result of this section.

**Theorem 1.** *Suppose strong feasibility condition holds. Let* $\eta \in (0,2)$, $\mathscr{B} \subset \mathscr{A}_\gamma$ *and* $\alpha$ *be defined as in Algorithm* 1, *and let* $K \doteq \lceil \|x_0 - x_*\|^2/\alpha^2\eta(2-\eta)\rceil$, *where* $x_*$ *is the center of* $\mathscr{B}$, *and* $\lceil x \rceil$ *indicates the smallest integer greater or equal than* $x$. *Then, with respect to Algorithm* 1, *the following results hold*:

(1) *If* $k_i$ *is chosen according to Rule* 1, *the Algorithm* 1 *terminates* (*finds a feasible solution and remains there*) *after a number of cycles less or equal than* $K$;
(2) *If* $k_i$ *is chosen according to Rule* 2, *then Algorithm* 1 *terminates after a number of iterations less or equal than* $K$;
(3) *If* $k_i$ *is chosen according to Rule* 3, *then*, *with probability one*, *Algorithm* 1 *terminates after a number of correction steps less or equal than* $K$.

**Proof.** First, if $m_{k_i}(x_i) > 0$, we rewrite correction step (11) in a more compact way as $x_{i+1} = \Pi_{\mathscr{Q}\gamma}[x_i - \lambda_i(\partial_x m_{k_i}(x_i)/\|\partial_x m_{k_i}(x_i)\|)]$, with the slight abuse of notation $\Pi_{\mathscr{Q}\gamma}[x] \doteq \binom{c}{\Pi_{\mathscr{Q}\gamma}[P]}$.

Secondly, we define

$$\bar{x} = \begin{pmatrix} \bar{c} \\ \bar{P} \end{pmatrix} = x_* + \alpha \frac{\partial_x m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|}. \quad (13)$$

Clearly, $\bar{x} \in \mathscr{B}$, and therefore is a feasible solution of (7) and, in particular, $m_k(\bar{x}) \leqslant 0$ for all $k = 1, \dots, N$ and $\operatorname{Tr} \bar{P} \geqslant \gamma$. Now, if $m_{k_i}(x_i) > 0$, it follows from Property 2 of Lemma 3 that

$$\rho_{i+1} \doteq \|x_{i+1} - x_*\|^2 \leqslant \left\| x_i - \lambda_i \frac{\partial_x m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|} - x_* \right\|^2$$
$$= \rho_i + \lambda_i^2 - 2\lambda_i (x_i - \bar{x})^{\mathrm{T}} \frac{\partial_x m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|}$$
$$-2\lambda_i (\bar{x} - x_*)^{\mathrm{T}} \frac{\partial_x m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|}.$$

Consider now the last two terms of the above inequality. From the convexity of $m_{k_i}(x_i)$ and the feasibility of $\bar{x}$, it follows that $(x_i - \bar{x})^{\mathrm{T}} \partial_x m_{k_i}(x_i) \geqslant m_{k_i}(x_i) - m_{k_i}(\bar{x}) \geqslant m_{k_i}(x_i)$ while, the definition of $\bar{x}$ given in (13) leads to $(\bar{x} - x_*)^{\mathrm{T}} (\partial_x m_{k_i}(x_i)/\|\partial_x m_{k_i}(x_i)\|) = \alpha$. Therefore, we write

$$\rho_{i+1} \leqslant \rho_i + \lambda_i^2 - 2\lambda_i \left( \frac{m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|} + \alpha \right).$$

Now, if $m_{k_i}(x_i) > 0$, substituting the value of $\lambda_i$ given in (12), we get

$$\rho_{i+1} \leqslant \rho_i - \eta(2-\eta) \left( \frac{m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|} + \alpha \right)^2$$
$$\leqslant \rho_i - \alpha^2 \eta (2-\eta).$$

From this formula, we conclude that no more than $K = \lceil \|x_0 - x_*\|^2/\alpha^2\eta(2-\eta)\rceil$ correction steps should be executed. Statement 1 of the theorem is then immediately proved considering that at each cycle Rule 1 guarantees to that, if $x_i \notin A_\gamma$, at least a violated constraint $m_{k_i}(x_i) > 0$ is found, and therefore at least a correction step is performed. Similarly, Statement 2 is proved verifying that Rule 2 considers at every step a violated constraint and in that way guarantees a correction step. Finally, to prove Statement 3, we notice that at each step Rule 3 associates to each constraint a probability strictly greater than zero. Thus, there is a nonzero probability to make a correction step and with probability one, the method cannot terminate at an infeasible point. We, therefore, conclude that the algorithm must terminate after a finite number of iterations at a solution in the set $x_i \in A_\gamma$. □

In the next section, we provide rigorous results for an algorithm explicitly tailored for optimization problem (6).

## 4. Direct optimization

The following algorithm embeds the quasi-projection and the function optimization steps. The algorithm is designed to converge directly to the optimal value $f_*$ of Problem 1.

**Algorithm 2.** *Consider a starting point* $x_0 = \binom{c_0}{P_0}$, $c_0 \in \mathbb{R}^n$, $P_0 \in \mathscr{Q}\gamma$ *and construct the following recursion*, *provided that Rule* 2 *is applied to choose* $k_i$ *and* $m(x_i)$ *denotes*

$m(x_{k_i}) = \max_k m_k(x_i)$:

- (constraint correction step) *if* $m(x_i) > 0$,

$$x_{i+1} = \begin{pmatrix} c_{i+1} \\ P_{i+1} \end{pmatrix} = \begin{pmatrix} c_i - \lambda_i \dfrac{\partial_c m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|} \\ \Pi_{\mathscr{C}}[P_i - \lambda_i \dfrac{\partial_P m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|}] \end{pmatrix},$$

*with stepsize*

$$\lambda_i = \left(\frac{m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|} + \mu_i\right), \tag{14}$$

- (functional correction step) *if* $m(x_i) \leqslant 0$,

$$x_{i+1} = \begin{pmatrix} c_{i+1} \\ P_{i+1} \end{pmatrix} = \begin{pmatrix} c_i + \lambda_i \dfrac{\nabla_c f(x_i)}{\|\nabla f(x_i)\|} \\ \Pi_{\mathscr{C}}[P_i + \lambda_i \dfrac{\nabla_P f(x_i)}{\|\nabla f(x_i)\|}] \end{pmatrix},$$

*with stepsize*

$$\lambda_i = \left(\frac{[F - f(x_i)]_+}{\|\nabla f(x_i)\|} + \mu_i\right), \tag{15}$$

*and* $F < f_*$. *The scalar sequence* $\mu_i > 0$ *is chosen such that* $\mu_i \to 0$, $\sum \mu_i = \infty$.

The intuition under Algorithm 2 is to proceed with a gradient descent towards the objective function $f(x)$ every time the current ellipsoid does not violate the considered constraint. This method has some common features with the method proposed in Polyak (1967). However, in that paper the stepsize was taken $\lambda_i = \mu_i$ (this leads to slower convergence than Algorithm 2) and only vector variables, rather than matrix, were considered.

The gradients of the function $f(x) = \operatorname{Tr} P$ are given by $\nabla_P f(x) = I$, $\nabla_c f(x) = 0$. The algorithm may be also applied to the case where $f(x) = \log\det P$. This function would lead to the gradient $\nabla_P f(x) = P^{-1}$, $\nabla_c f(x) = 0$. The parameter $F$ could in principle be chosen as $F = 0$, but, to improve convergence, we suggest the choice $F = \max\{f(x_k)\colon k \leqslant i,\ m(x_i) \leqslant 0\}$. For Algorithm 2, the following result on asymptotic convergence holds.

**Theorem 2.** *If the sequence* $x_i$ *is generated by Algorithm* 2, *then there exists an admissible subsequence* $x_{i_t}$: $m(x_{i_t}) \leqslant 0$ *such that* $\lim_{t\to\infty} f(x_{i_t}) = f_*$.

**Proof.** Choose any $\hat{f} < f_*$; if we can prove that Algorithm 2 solves a feasibility problem

Find $x{:}P \in \mathscr{C}$,

$$m_k(x) \leqslant 0, \quad k = 1, \ldots, N, \tag{16}$$
$$f(x) \geqslant \hat{f}$$

for all $\hat{f} < f_*$ in a finite number of iterations, this is equivalent to the statement of the theorem. Due to assumption on $\mathscr{D}$, the feasible set of (16) has an interior point; denote it $x_*$ while the radius of a ball centered in $x_*$ and contained in (16) will be denoted as $\alpha$. We proceed by writing the distance of the current point $x_{i+1}$ from $x_*$: $\rho_{i+1} = \|x_{i+1} - x_*\|^2$.

In the case of a constraint correction step, i.e. $m(x_i) > 0$, as in the proof of Theorem 1 we write $\rho_{i+1} \leqslant \rho_i + \lambda_i^2 - 2\lambda_i(\frac{m_{k_i}(x_i)}{\|\partial_x m_{k_i}(x_i)\|} + \alpha)$. Then, substituting the value of $\lambda_i$ given in (14) we get $\rho_{i+1} \leqslant \rho_i + \mu_i^2 - 2\mu_i\alpha$. For $i$ large enough $\mu_i < \alpha$, thus

$$\rho_{i+1} \leqslant \rho_i - \mu_i\alpha. \tag{17}$$

Consider now the case of functional correction step. We define $\bar{x} = x_* - \alpha\nabla f(x_i)/\|\nabla f(x_i)\|$. Again, following a similar derivation as in the proof of Theorem 1, we can obtain the estimate

$$\rho_{i+1} \leqslant \rho_i + \lambda_i^2 - 2\lambda_i\left(\frac{\hat{f} - f(x_i)}{\|\nabla f(x_i)\|} + \alpha\right). \tag{18}$$

Now two situations can be met.

(a) $f(x_i) \geqslant F$. Then $\lambda_i = \mu_i$ and (18) becomes $\rho_{i+1} \leqslant \rho_i + \mu_i^2 - 2\mu_i(\frac{\hat{f} - f(x_i)}{\|\nabla f(x_i)\|} + \alpha)$. We assume that $f(x_i) \leqslant \hat{f}$ (otherwise $x_i$ is a solution of (16)); then from above inequality we arrive to (17).

(b) $f(x_i) < F$. Then $\lambda_i = (F - f(x_i))/\|\nabla f(x_i)\| + \mu_i$ and we may assume $\mu_i \leqslant \alpha$ (i.e. $i$ is large enough) and $F \leqslant \hat{f}$ (i.e. $\hat{f}$ is close enough to $f_*$). Under these conditions (18) implies

$$\begin{aligned}
\rho_{i+1} &\leqslant \rho_i + \left(\frac{F - f(x_i)}{\|\nabla f(x_i)\|} + \mu_i\right)^2 \\
&\quad -2\left(\frac{F - f(x_i)}{\|\nabla f(x_i)\|} + \mu_i\right)\left(\frac{\hat{f} - f(x_i)}{\|\nabla f(x_i)\|} + \alpha\right) \\
&\leqslant \rho_i - \left(\frac{F - f(x_i)}{\|\nabla f(x_i)\|} + \mu_i\right)\left(\frac{\hat{f} - f(x_i)}{\|\nabla f(x_i)\|} + \alpha\right) \\
&\leqslant \rho_i - \alpha\mu_i.
\end{aligned}$$

Thus in all cases we get the same inequality (17). Such inequality can hold just a finite number of steps, because $\sum \mu_i = \infty$. Hence, after a finite number of iterations we arrive to the set defined by (16). □

As already remarked, while the feasibility algorithm presented in Section 3 has some common features with the techniques presented in Calafiore and Polyak (2001) in the field of robust LMI, the optimization algorithm presented here is, to our best knowledge, a new result.

## 5. Complexity issues

Now let us compare the proposed algorithms with known ones and discuss their numerical complexity. The problem of finding the largest inscribed ellipsoid (1) or feasibility problem (7) have been considered in Nesterov and Nemirovskii (1994), Vandenberghe and Boyd (1999) and Khachiyan and Todd (1990). In these papers, the feasibility problem is rewritten as a system of LMIs, where constraints $m_k(x) \leqslant 0,\ k = 1,\dots,N$ look like

$$\begin{pmatrix} (b_k - a_k^{\mathrm{T}} c)I & Pa_k \\ a_k^{\mathrm{T}} P & b_k - a_k^{\mathrm{T}} c \end{pmatrix} \succcurlyeq 0, \quad k = 1,\dots,N.$$

Thus, we have $N$ LMIs, each one of dimension $(n+1)\times(n+1)$. In typical problems (see Section 6) $N$ is large enough, and the simultaneous solution of such inequalities via standard tools (e.g. LMI-toolbox in MATLAB) is time-consuming or impossible.

Specialized techniques for solving the feasibility problem are presented in Nesterov and Nemirovskii (1994) and Khachiyan and Todd (1990). Both algorithms require $CN^{3.5}\ln N$ arithmetic operations, where $C$ depends on $n$, on radius of the ball inscribed in $\mathscr{D}$ and on $\gamma$. Again, for $N$ large their complexity may be too high.

Another approach to solution of the largest inscribed ellipsoid problem is to handle it by methods of second-order cone programming (see Saigal, Vanderberghe, & Wolkowicz, 2000) such as Sedumi or SDPack. Large dimension of $N$ makes the methods of this sort noneffective.

Sometimes the opinion is stated that the large number of constraints $N$ is not a serious obstacle, because most of them are redundant and do not work to generate the boundary of the admissible polytope $\mathscr{D}$. Thus, after removing them the true number of constraints is much smaller. However such "refinement" of constraints is rather expensive since it requires multiple solution of linear programming problems.

It is of interest to estimate the numerical complexity of the above proposed algorithms. At each iteration we deal with one inequality only. Calculation of the subgradient requires $O(n^2)$ arithmetic operations (see Lemma 2) while calculation of the projection on $\mathscr{C}$ or $\mathscr{Q}_\gamma$ costs $O(n^3)$ operations. Each cycle for Rule 1 requires $N$ iterations and one iteration for Rule 2 of Algorithm 1 consists of processing of $N$ inequalities, while the number of cycles or iterations depends just on geometry of $\mathscr{D}$ (see Theorem 1). We conclude that Algorithm 1 finds a solution of the feasibility problem in $O(Nn^3)$ arithmetic operations. For moderate $n$ and large $N$ this is much better than the estimate $C(n)N^{3.5}$ for algorithms of Nesterov and Nemirovskii (1994) and Khachiyan and Todd (1990).

We also remark that another obvious but important advantage of the proposed algorithm, compared to interior-point methods is its recursive nature: Even if the complexity of both methods is linear in $N$, the algorithm is important in situations where a recursive method is desirable. This is the case for example of the set membership setting discussed in the next section.

## 6. Application to set membership identification

In this section, the general results derived in Sections 3 and 4 are specialized to the approximation of the set of feasible parameter arising in set membership identification. Formally, we consider a single-output linear regression model,

$$y_k = a_k^{\mathrm{T}}\theta + e_k, \quad k = 1,\dots,N, \tag{19}$$

where $\theta \in \mathbb{R}^n$ is the parameter vector and $a_k^{\mathrm{T}} \in \mathbb{R}^n$ is the regressor. The measurement error $e_k$, that includes measurement noise and modelling error, is assumed to be unknown but bounded, that is $|e_k| \leqslant \bar{e}$. The error bound $\bar{e}$ is assumed to be known a priori. Each inequality of type (19) defines a strip $\mathscr{S}_k$ in the parameter space

$$\mathscr{S}_k \doteq \{\theta \in \mathbb{R}^n\colon\ y_k - \bar{e} \leqslant a_k^{\mathrm{T}}\theta \leqslant y_k + \bar{e}\}. \tag{20}$$

This leads to the determination of a *membership set* $\mathscr{D}$, defined as the set of parameters $\theta$ that are consistent with the measurements, the assumed model structure and the a priori error bounds. This polytope is represented by the intersection of the strips

$$\mathscr{D} \doteq \{\theta \in \mathbb{R}^n\colon\ |y_k - a_k^{\mathrm{T}}\theta| \leqslant \bar{e}, k = 1,\dots,N\} = \bigcap_{k=1,\dots,N} \mathscr{S}_k.$$

In this setup, features extraction from the polytope has been performed approximating the polytope for instance by means of outer bounding ellipsoids, see Belforte, Bona, and Cerone (1990), Fogel and Huang (1982) and Pronzato and Walter (1996). Outer bounding approximations are guaranteed to contain all possible feasible parameters. However, this choice in general carries over an high degree of conservatism, since the outer bounding set can include points that are not compatible with the adopted model and the measurements. This can be a major drawback, especially if one is interested in the estimation of a nominal parameter. For this reason, in the last years, inner approximating sets have been introduced. Moreover, the size of such set can be of primary interest for control because it provides a lower bound on the possible model uncertainty. This motivates the interest on extending our result to set membership identification.

The conditions for the ellipsoid $\mathscr{E}$ to belong to the strip $\mathscr{S}_k$ can be derived directly from Lemma 1: the ellipsoid $\mathscr{E}$ defined in (3) belongs to $\mathscr{S}_k$ given in (20) if and only if the following condition holds:

$$\tilde{m}_k(x) \leqslant 0, \tag{21}$$

where $\tilde{m}_k(x) \doteq |y_k - a_k^{\mathrm{T}}c| + \|Pa_k\| - \bar{e}$.

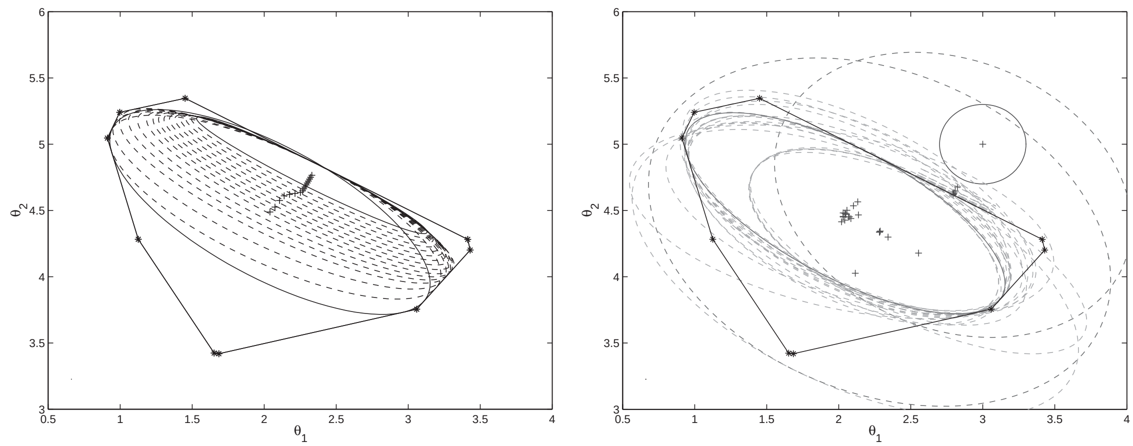


Fig. 1. Application of Algorithms 1 and 2 to Example 1.

Algorithms 1 and 2 can then be specialized to directly handle this particular case, simply replacing $m_k(x)$ with $\tilde{m}_k(x)$ and considering that the subgradients of this new function are given by

$$\partial_x \tilde{m}_{k_i}(x_i) = \begin{pmatrix} \partial_c \tilde{m}_{k_i}(x_i) \\ \partial_P \tilde{m}_{k_i}(x_i) \end{pmatrix} = \begin{pmatrix} a_{k_i} \operatorname{sgn}(a_{k_i}^{\mathrm{T}} c - y_{k_i}) \\ \dfrac{a_{k_i} a_{k_i}^{\mathrm{T}} P + P a_{k_i} a_{k_i}^{\mathrm{T}}}{2\|P a_{k_i}\|} \end{pmatrix}.$$

This leads to $\|\partial_x \tilde{m}_{k_i}(x_i)\|^2 = 2\|a_{k_i}\|^2$.

We remark that, the center of the ellipsoid obtained by Algorithm 1 provides an estimate of the parameter in a finite number of iterations. Moreover, an important information about the reliability of the estimate is given by the size (and the shape) of the ellipsoid. In this sense, this result extends previous works on successive projections methods for set membership identification (see for instance Bai, Cho, & Tempo, 1998).

Note that typically the number of measurements is relatively large if compared with the dimension of parameters (say, $n \approx 10, N \approx 1000$). Thus, the considerations of Section 5 confirm that the proposed methods are well tailored for this problem.

## 7. Examples

**Example 1.** For visualization purposes, we provide here a two-dimensional example. We considered a set of 50 random data constructed as follows: for $k = 1, \dots, 50$ we set $a_k^{\mathrm{T}} = [1\ v_1]$ and $b_k = v_2$, where $v_1,\ v_2$ are uniform real numbers in the interval $[-1, 1]$. In order to compare the result, we applied the different algorithms presented using the same initial ellipsoid with parameters $c_0 = [3\ 5]^{\mathrm{T}}$, $P_0 = \mathrm{diag}(0.3, 0.3)$. The results are reported in Fig. 1, together with the convex polytope computed applying the techniques given in Mo and Norton (1990). As a general comment on the implementation of the algorithm, we observed that the parameter $\eta$ turns out to be quite important to speed up the algorithm convergence. In fact, whereas the choice of $\eta = 1$ is optimal in the worst-case, simulation showed a general better performance of the algorithm for values of $\eta$ around 1.5. Intuitively, this corresponds to slightly "push" at every step the projection inside the constraint.

In Fig. 1a, we present an iterative application of Algorithm 1 with update Rule 3 and increasing values of $\gamma$ (200 steps for each $\gamma$, with $\eta = 1.5$ and $\alpha = 0.01$). Dotted lines represent solutions for $\gamma$ starting from 1 and increasing at steps of 0.05. The stopping rule was $m_{k_{200}}(x_{200}) \geqslant 10^{-5}$, and we adopted as a solution the outcome of the previous run of the algorithm. The final ellipsoid (solid line) was obtained for $\gamma = 1.75$.

Fig. 1b presents the application of Algorithm 2 to the same example, with same initial conditions and $\eta = 1.5$, $\mu_i = i^{-0.95}$. After 300 iterations (284 *constraint correction* steps and 16 *functional correction* steps), we obtained the ellipsoid (solid line) with $\mathrm{Tr}\, P_{300} = 1.747$, which is perfectly consistent with the solution reported in Fig. 1a.

**Example 2.** In this second example we considered an identification problem taken from Pronzato and Walter (1996) (see Example 3, p. 133). We considered an AR-5 model described by

$$y(k) = -0.4y(k-1) - 0.85y(k-2) - 0.1y(k-3) \\ -0.02y(k-4) - 0.05y(k-5) + e(k) \qquad (22)$$

for $k = 6, \dots, N$ and initial values $y(k) = e(k)$, for $k = 1, \dots, 6$. For simulation purposes the measurement error $e(k)$ was considered uniformly distributed in the interval $[-1, +1]$ and $N = 200$ random values of $e(k)$ have been generated. The modified version of Algorithm 2 presented in Section 6 was then applied to this set of data. The parameters of the algorithm where set to $\eta = 1.5$ and $\alpha = 10^{-6}$. The final solution was

$$c = \begin{bmatrix} 0.4236 \\ 0.9081 \\ 0.1936 \\ 0.1318 \\ 0.1136 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.0204 & 0.0032 & 0.0085 & -0.0201 & 0.0019 \\ 0.0032 & 0.0530 & 0.0308 & 0.0517 & 0.0381 \\ 0.0085 & 0.0308 & 0.0618 & 0.0672 & 0.0411 \\ -0.0201 & 0.0517 & 0.0672 & 0.1421 & 0.0449 \\ 0.0019 & 0.0381 & 0.0411 & 0.0449 & 0.0558 \end{bmatrix}.$$

The trace of this result is $\mathrm{Tr}(P) = 0.0331$, and the volume is $3.7494e - 008$.

## 8. Conclusions and future directions

In this paper, we presented two recursive algorithms for constructing an inner ellipsoidal approximation of a polytope described by linear inequalities. The first one solves a feasibility problem, that is it searches for an ellipsoid of given size contained in the polytope $\mathscr{D}$, while the second directly tackles the optimization problem of finding the largest ellipsoid in $\mathscr{D}$. The main idea underneath both algorithms is to process in a iterative way the constraints describing the polytope. This allows to overcome the computational and memory storage problems that usually arise whenever standard methods are used to treat problems with large number of constraints. We then specialized our results to the particular structures arising in set membership identification.

In concluding this paper, we would like to observe that the techniques presented here can also be extended to deal with specific problems arising in different fields such as pattern recognition, discriminant analysis, cluster analysis. As a simple example, let us consider a classical discrimination problem presented for instance in Kendall and Stewart (1977), Chapter 44: *Given two sets $\Theta_1$, $\Theta_2$ in $\mathbb{R}^n$, find a linear function which is positive on $\Theta_1$ and negative on $\Theta_2$*. Recent extensions to quadratic functions and in particular ellipsoids have been proposed, see for instance Boyd et al. (1994). This kind of problems may be solved using an approach similar to the one presented here.

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