Mixed Deterministic/Randomized Methods for Fixed Order Controller Design

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***Abstract*—In this paper, we propose a general methodology for designing fixed order controllers for single-input single-output plants. The controller parameters are classified into two classes: randomized and deterministically designed. For the first class, we study randomized algorithms. In particular, we present two low-complexity algorithms based on the Chernoff bound and on a related bound (often called "log-over-log" bound) which is generally used for optimization problems. Secondly, for the deterministically designed parameters, we reformulate the original problem as a set of linear equations. Then, we develop a technique which efficiently solves it using a combination of matrix inversions and sensitivity methods. A detailed complexity analysis of this technique is carried on, showing its superiority (from the computational point of view) to existing algorithms based on linear programming. In the second part of the paper, these results are extended to H_∞ performance. One of the contributions is to prove that the deterministically designed parameters enjoy a special convex characterization. This characterization is then exploited in order to design fixed order controllers efficiently. We then show further extensions of these methods for stabilization of interval plants. In particular, we derive a simple one-parameter formula for computing the so-called critical frequencies which are required by the algorithms.**

***Index Terms*—Fixed order controller design, H_∞ performance, interval plants, randomized algorithms, stabilization.**

I. Introduction

In recent years, research within systems and control focused on *computationally difficult* problems such as designing a fixed order output feedback controller without any specific structural assumption. This problem is known to share similar difficulties with the well-known static output feedback design, see [1]. The source of these computational difficulties lie in the NP-hardness (see [2] for a discussion of complexity issues of fixed order output feedback design), which is essentially unavoidable with an approach fully deterministic. The interested reader is referred to [3]–[5] for recent developments regarding fixed order controller design.

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For controllers with very special structure such as PID or lead-lag, see, e.g., [6]–[8], a number of useful methods are now available for the control engineer. In particular, in the last few years, various innovative techniques for designing low order controllers have been proposed. These methods make use of a geometric characterization of the set of stabilizing PID gains. Subsequently, the controller is determined either by means of linear programming, see, e.g., [7], or with the aid of graphical methods, see e.g., [9]–[11].

On the other hand, an approach completely based on randomization is developed in [12] for static output feedback. This method is applicable to the problem under attention. However, the direct application is problematic in practice because the probability measure of feasible controllers is often small and its estimation may require a large number of samples. Moreover, not all the parameters need to be randomized due to the special structure of the problem, and this is a clear advantage from the deterministic point of view.

In this paper, we develop a general methodology to design fixed order stabilizing controllers of single-input single-output plants. The proposed approach is based on the idea of splitting the controller parameters into two classes, the so-called *randomized parameters* and *deterministically designed parameters*. The latter will be referred to as *deterministic parameters* in short. We use a randomized algorithm for design of randomized parameters, which enables us to design not only PID or lead-lag controllers but also general fixed-order controllers. On the other hand, we use a deterministic technique for design of deterministic parameters. This takes into account the problem structure and leads to efficient algorithms. A precise description of these classes is given in Section II together with the notation used in the paper.

In Section III, we study randomized algorithms for determining the randomized parameters. Two specific randomized algorithms, based on the classical Chernoff bound, see [13], and on a so-called "log-over-log" bound, see [14], are presented. We recall that, while the former bound deals with probability estimation, the latter may be used for optimization, and it is related to the *fpras* (fully polynomial randomized approximation scheme) theory, see e.g., [15] for additional details. These randomized algorithms are low-complexity and easily implementable, but, of course, the drawback is to obtain a solution only with high probability. See [16] for a complete treatment of randomized algorithms for systems and control and [17] for discussions on statistical learning theory.

In Sections IV and V, as a counterpart to these random methods, we present an approach for determining the deterministic parameters. In particular, in Section IV, we study the geometry of the set of stabilizing controller coefficients, showing that this set is the (finite) union of polyhedral sets. This characterization is in fact a generalization of results obtained in various papers focused on low-order controllers such as PID or lead-lag, see Remark 1 in Section IV. Then, in Section V, we derive efficient techniques for computing the (deterministic) controller parameters. As an alternative to existing methods which use linear programming, we develop algorithms based on the solution of a set of linear equations by matrix inversion. Roughly speaking, these algorithms deal with vertices of stabilizing polyhedral sets rather than inequalities describing the polyhedral sets, which are instead used by linear programming. A detailed complexity analysis is carried on, showing that the vertex approach proposed here has polynomial-time complexity. The outcome of this method is to provide a (candidate) marginal stabilizer, so that a stabilizing controller can be easily obtained by means of standard sensitivity methods.

In Section VI, we extend these results to H_∞ performance of weighted sensitivity and complementary sensitivity functions. To this end, we extend the geometric characterization previously obtained for stabilization. In this case, the set of (deterministic) controller parameters satisfying a given bound on the H_∞ norm of the sensitivity function is no longer polyhedral, but it is shown to be the (infinite) intersection of sets $\Gamma(\phi)$, and, for fixed ϕ, each set $\Gamma(\phi)$ is the union of a finite number of polyhedral sets. This result is a generalization of previous results obtained for controllers having a special structure, see e.g., [11], [18], and [19]. In Section VII, we derive polynomial-time algorithms similar to those previously obtained for stabilization. However, the extension is not straightforward, and the main difference with the algorithms developed in Section V is the presence of a sweeping parameter ϕ bounded in the interval $[0, 2\pi)$. In this section, we also demonstrate a technical result, involving trigonometric functions of ϕ, which provides the so-called critical frequencies which are used in the algorithm.

In Section VIII, we establish related results for the case when a fixed plant is replaced by an interval plant. The main contribution is to show that the techniques previously developed can be suitably generalized to solve this problem, provided that an additional parameter $\lambda \in [0, 1]$ is introduced. In this case, the critical frequencies are given by the solution of two independent bivariate polynomial equations involving λ and the frequency. We remark that λ enters affinely in the first equation and quadratically in the second.

In Section IX, we study two application examples. We first illustrate the main ideas of this paper through stabilization. Then, we present an example of H_∞ performance design by the proposed algorithm. Section X summarizes the conclusions.

II. Preliminaries and Notation

We now introduce the notation used in this paper. Consider a single-input single-output strictly proper plant of the form

$$P(s) = \frac{N_P(s)}{D_P(s)} \tag{1}$$

where $N_P(s)$ and $D_P(s)$ are numerator and denominator plant polynomials of order n_N and n_D, respectively. We study a fixed order controller of the form

$$C(s) = \frac{N_C(s)}{D_C(s)}$$

where $N_C(s)$ and $D_C(s)$ are numerator and denominator controller polynomials, respectively. Without loss of generality, we rewrite $C(s)$ as

$$C(s) = \frac{X(s^2) + sY(s^2)}{Z(s^2) + sV(s^2)} \tag{2}$$

where $X(s^2)$, $Y(s^2)$, $Z(s^2)$ and $V(s^2)$ are polynomials containing only even powers of s. These polynomials are of the form

$$\begin{aligned} X(s^2) &= \theta_0 + \theta_2 s^2 + \theta_4 s^4 + \cdots + \theta_{n_X} s^{n_X} \\ Y(s^2) &= \alpha_0 + \alpha_2 s^2 + \alpha_4 s^4 + \cdots + \alpha_{n_Y} s^{n_Y} \\ Z(s^2) &= \beta_0 + \beta_2 s^2 + \beta_4 s^4 + \cdots + \beta_{n_Z} s^{n_Z} \\ V(s^2) &= \mu_0 + \mu_2 s^2 + \mu_4 s^4 + \cdots + \mu_{n_V} s^{n_V} \end{aligned} \tag{3}$$

and their orders in s are denoted by n_X, n_Y, n_Z and n_V, respectively.

We now classify the controller parameters into randomized and deterministic parameters, see Section III for further discussions. Four possibilities are considered in the classification, so that some flexibility is allowed in the design of fixed order stabilizing controllers. For example, the coefficients of $X(s^2)$, or those of $Y(s^2)$, may be chosen as deterministic parameters. Once this choice is made, the randomized parameters are set, so that they are the coefficients of $Y(s^2)$, $Z(s^2)$, $V(s^2)$, or those of the polynomials $X(s^2)$, $Z(s^2)$, $V(s^2)$. It will be shown later in Section V that a deterministic parameter value can be found in a computationally efficient way once the values of randomized parameters are computed as described in Section III. The following is the list of the four possible cases.

1) The deterministic parameters are the coefficients of $X(s^2)$ and the randomized parameters are the coefficients of $Y(s^2)$, $Z(s^2)$, $V(s^2)$;
2) The deterministic parameters are the coefficients of $Y(s^2)$ and the randomized parameters are the coefficients of $X(s^2)$, $Z(s^2)$, $V(s^2)$;
3) The deterministic parameters are the coefficients of $Z(s^2)$ and the randomized parameters are the coefficients of $X(s^2)$, $Y(s^2)$, $V(s^2)$;
4) The deterministic parameters are the coefficients of $V(s^2)$ and the randomized parameters are the coefficients of $X(s^2)$, $Y(s^2)$, $Z(s^2)$.

Let n_θ and n_η be the number of deterministic and randomized parameters and θ and η be the vectors containing the values of the deterministic and randomized parameters, respectively. We write as Θ and $\mathcal{N}$ the sets of possible values of θ and η and, without loss of generality, we assume that they are bounded. For simplicity, in the rest of this paper, we consider only the first case in the above list. That is, the deterministic parameters are the coefficients of $X(s^2)$ and the randomized parameters are the coefficients of $Y(s^2)$, $Z(s^2)$ and $V(s^2)$. This implies that

$$\begin{aligned} n_\theta &= \frac{n_X}{2} + 1 \\ n_\eta &= \frac{n_Y + n_Z + n_V}{2} + 3 \\ \theta &= [\theta_0 \quad \cdots \quad \theta_{n_X}]^T \\ \eta &= [\alpha_0 \quad \cdots \quad \alpha_{n_Y} \quad \beta_0 \quad \cdots \quad \beta_{n_Z} \quad \mu_0 \quad \cdots \quad \mu_{n_V}]^T. \end{aligned}$$

The other three cases can be treated in a similar manner.

The closed-loop polynomial $p(s)$ of negative feedback connection consisting of a plant $P(s)$ and a controller $C(s)$ is given by the equation

$$\begin{aligned} p(s) &= N_P(s)N_C(s) + D_P(s)D_C(s) \\ &= N_P(s)(X(s^2) + sY(s^2)) + D_P(s)(Z(s^2) + sV(s^2)) \end{aligned}$$

where the degree of $p(s)$ is assumed to be fixed. That is, we consider the generic subset of parameters of the controller coefficients which does not change the degree of the closed-loop polynomial. The first objective of this paper is stabilization. This means finding controller parameters so that the closed-loop polynomial $p(s, \theta, \eta)$ has all its zeros in the open left half plane; i.e., it is stable. If a controller $C(s, \theta, \eta)$ is determined, we call it a *stabilizing controller*.

III. Randomized Algorithms for Computing Controller Parameters

In Section IV, we will show that, if the values of randomized parameters are fixed, the set of deterministic parameter values corresponding to stabilizing controllers enjoys some convexity property which can be exploited for efficient computation of the deterministic parameter values. For the randomized parameter values, however, no convexity is known and their efficient computation is difficult deterministically. In order to overcome this difficulty, we consider in this section the use of randomized algorithms, which are known to be effective for many deterministically difficult problems within systems and control, see, e.g., [16], [20]–[23].

We say that a randomized parameter value η is *feasible* if there exists $\theta \in \Theta$ such that $C(s, \theta, \eta)$ is a stabilizing controller. Here, we present a randomized algorithm to find such a value η. For this purpose, we assign a probability distribution $\mathcal{P}$ to the set $\mathcal{N} \subseteq \mathbf{R}^{n_\eta}$, which is the bounding set of parameters η. Let ϵ and δ be any positive numbers less than unity and define

$$N_1 = \left\lceil \frac{\ln(1/\delta)}{\ln(1/(1-\epsilon))} \right\rceil$$

where $\ln$ denotes the natural logarithm and $\lceil x \rceil$ denotes the smallest integer which is greater than or equal to x. Now we propose the following algorithm.

Algorithm 1

1. for $i := 1, \ldots, N_1$ do
   begin
2. draw a sample $\eta^{(i)} \in \mathcal{N}$ according to $\mathcal{P}$;
3. if $\eta^{(i)}$ is feasible then return;
   end

We observe that the feasibility check of Step 3 requires $\eta^{(i)}$ generated at Step 2 and a suitable θ which may be provided by the (deterministic) Algorithm 3 presented in Section V. The performance of the algorithm above is guaranteed by the following theorem, which is an immediate consequence of the results in [14]. In particular, this algorithm gives a feasible η with high probability unless the set of feasible η's is too small. Here, we let $\mathcal{A}$ denote the set of all feasible η in $\mathcal{N}$ and $\mathcal{P}(\mathcal{A})$ its measure according to $\mathcal{P}$. An estimate of $\mathcal{P}(\mathcal{A})$ is provided in Theorem 2.

Theorem 1: Suppose that the measure $\mathcal{P}(\mathcal{A})$ is greater than ϵ. Then, the probability that no $\eta^{(i)}$, $i = 1, \ldots, N_1$, provided by Algorithm 1 is feasible is less than δ.

Note that the maximum number of samples N_1 depends only on ϵ and δ. As we will see in Section IV, Line 3 of the algorithm can be carried out in polynomial time. Although the complexity to execute Line 2 depends on $\mathcal{N}$ and $\mathcal{P}$, it is usually polynomial in the dimension n_η; for example, when $\mathcal{N}$ is box-shaped and $\mathcal{P}$ is the uniform distribution. This choice is used in many practical applications, see, e.g., [16] for further discussions.

Next, we present another randomized algorithm to evaluate the measure $\mathcal{P}(\mathcal{A})$. We choose positive numbers ϵ and δ to be smaller than unity and define

$$N_2 = \left\lceil \frac{1}{2\epsilon^2} \ln \frac{2}{\delta} \right\rceil.$$

Algorithm 2

1. set $N_s := 0$;
2. for $i := 1, \ldots, N_2$ do
   begin
3. draw a sample $\eta^{(i)} \in \mathcal{N}$ according to $\mathcal{P}$;
4. if $\eta^{(i)}$ is feasible then set $N_s := N_s + 1$;
   end

Here, N_s counts the number of feasible $\eta^{(i)}$ among the N_2 samples. We have the following theorem based on the well-known Chernoff bound [13], [16].

Theorem 2: The probability that $|N_s/N_2 - \mathcal{P}(\mathcal{A})| > \epsilon$ holds is less than δ.

IV. Polyhedral Characterization for Stabilization

In this section, we characterize the set of deterministic parameters which stabilize the plant. This enables us to develop an efficient deterministic algorithm to handle the deterministic parameters.

The first step is to study the so-called critical frequencies. Let us consider the closed-loop polynomial

$$p(s) = p_0(s) + p_1(s)X(s^2)$$

where

$$\begin{aligned} p_0(s) &= sN_P(s)Y(s^2) + D_P(s)(Z(s^2) + sV(s^2)) \\ p_1(s) &= N_P(s). \end{aligned}$$

To express more explicitly the controller parameters θ, we write

$$p(s,\theta)=p_0(s)+p_1(s)[\theta_0+\theta_2 s^2+\cdots+\theta_{2(n_\theta-1)}s^{2(n_\theta-1)}] \quad (4)$$

where for simplicity we neglect the dependence on η.

Suppose that the deterministic parameters θ are on the boundary of the desired stabilizing set. Then, the system is marginally stable and satisfies

$$\begin{aligned}0&=p(j\omega,\theta)\\ &=p_0(j\omega)\\ &\quad+p_1(j\omega)[\theta_0+\theta_2(j\omega)^2+\cdots+\theta_{2(n_\theta-1)}(j\omega)^{2(n_\theta-1)}] \quad (5)\end{aligned}$$

for a real number ω.[1] Notice that the factor $\theta_0+\theta_2(j\omega)^2+\cdots+\theta_{2(n_\theta-1)}(j\omega)^{2(n_\theta-1)}$ takes only real values while $p_0(j\omega)$ and $p_1(j\omega)$ can take complex values. Hence, only at some frequency ω, there exists θ satisfying (5). This motivates us to introduce the following notion.

Definition 1: A real number ω satisfying (5) for some θ is called a *critical frequency*.

The critical frequencies ω are classified into the following three classes.

1) $\omega=0$. In this case, (5) is reduced to $0=p_0(0)+p_1(0)\theta_0$. This situation occurs only when $p_1(0)\neq 0$.
2) $\omega=\infty$. This situation occurs only when the degree of $p_0(s)$ is less than or equal to the degree of $p_1(s)X(s^2)$.
3) ω is a finite nonzero frequency. This situation occurs only when ω satisfies

$$I_0(-\omega^2)R_1(-\omega^2)-I_1(-\omega^2)R_0(-\omega^2)=0 \quad (6)$$

as well as $p_1(j\omega)\neq 0$. Here, we write $p_0(s)=R_0(s^2)+sI_0(s^2)$ and $p_1(s)=R_1(s^2)+sI_1(s^2)$ separating even and odd terms. Note that the number of such ω's is finite.

We let Ω denote the set of distinct critical frequencies with nonnegative signs. We let n_f denote the cardinality of Ω. The next lemma states that n_f is bounded by a polynomial of n_N, n_D, n_Y, n_Z and n_V and is independent of n_θ, i.e., n_X.

Lemma 1: The number of critical frequencies n_f is bounded as

$$n_f\leq\frac{\deg p_0(s)\deg p_1(s)-\min\{\deg p_0(s),\deg p_1(s)\}}{2}+2$$

where $\deg p_0(s)$ and $\deg p_1(s)$ are the degree of the polynomials $p_0(s)$ and $p_1(s)$ which are given by

$$\begin{aligned}\deg p_0(s)&=\max\{n_N+n_Y+1,n_D+\max\{n_Z,n_V+1\}\}\\ \deg p_1(s)&=n_N.\end{aligned}$$

Proof: Direct calculation gives the lemma. ■

For each critical frequency $\omega_i\in\Omega$, we obtain a hyperplane of the form

$$\psi^T(\omega_i)\theta=\nu(\omega_i) \quad (7)$$

[1]We remark that $j\omega$ may be a multiple root.

where

$$\begin{aligned}\psi^T(\omega_i)&=\begin{bmatrix}1 & -\omega_i^2 & \omega_i^4 & \cdots & (-1)^{n_\theta-1}\omega_i^{2(n_\theta-1)}\end{bmatrix}\\ \nu(\omega_i)&=-\frac{p_0(j\omega_i)}{p_1(j\omega_i)}.\end{aligned}$$

In the special case of $\omega_i=\infty$, the definitions are replaced by

$$\psi^T(\omega_i)=\begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix},\nu(\omega_i)=-\lim_{\omega\to\infty}\frac{p_0(j\omega)}{p_1(j\omega)}.$$

We now state a result which characterizes a stabilizing θ.

Theorem 3: Suppose that a randomized parameter vector η is selected according to Algorithm 1. Then, the set of all deterministic parameter vectors θ that stabilize the plant is either empty or is a union of a finite number of polyhedral sets.

Proof: By the preceding discussion, the set of stabilizing θ's has the boundary within the union of a finite number of hyperplanes. This gives the statement of the theorem. ■

This result studies stabilization properties in controller coefficients space for deterministic parameters. Since we consider the case when the coefficients of $Y(s^2)$, $Z(s^2)$ and $V(s^2)$ are fixed, the result says that the set of all deterministic stabilizing controller parameters of the form

$$C(s,\theta)=\frac{\theta_0+\theta_2 s^2+\cdots+\theta_{2(n_\theta-1)}s^{2(n_\theta-1)}+sY(s^2)}{Z(s^2)+sV(s^2)}$$

is a finite union of polyhedral sets, provided that a stabilizing controller exists. In other words, the "deterministic parameters" of $X(s^2)$ enjoy a polyhedral property which may be exploited in the development of computational methods. In fact, specific stabilizing controllers and the set of all stabilizing controllers can be obtained, in principle, by solving a number of linear programs. However, since (7) is a linear equation, in order to study stabilizing regions, all possible combinations of the inequalities

$$\psi^T(\omega_i)\theta\geq\nu(\omega_i)$$

and

$$\psi^T(\omega_i)\theta\leq\nu(\omega_i)$$

for $\omega_i\in\Omega$ should be considered as constraints of the linear program. In turn, this leads to an exponential number of linear programs.

To see this more precisely, let us introduce matrices

$$\Psi\doteq\begin{bmatrix}\psi^T(\omega_1)\\ \vdots\\ \psi^T(\omega_{n_f})\end{bmatrix},\qquad \nu\doteq\begin{bmatrix}\nu(\omega_1)\\ \vdots\\ \nu(\omega_{n_f})\end{bmatrix}.$$

Note that $\Psi\in\mathbf{R}^{n_f\times n_\theta}$ and $\nu\in\mathbf{R}^{n_f}$. We also consider a diagonal matrix $S_i\in\mathbf{R}^{n_f\times n_f}$ with diagonal elements either equal to -1 or 1. We see that the total number of such S_i is 2^{n_f}. Thus, all possible deterministic parameter regions generated by the critical frequencies are characterized as

$$S_i\Psi\theta\geq S_i\nu,\qquad i=1,2,\ldots,2^{n_f}.$$

We therefore conclude that the total number $N_{\mathrm{LP}}(n_f)$ of required linear programs in the worst-case is given by

$$N_{\mathrm{LP}}(n_f) = 2^{n_f}.$$

Motivated by this discussion, in the next section we present a more efficient and direct procedure, which reduces the complexity of the algorithm and avoids the combinatoric explosion in the number of linear programs. The first observation we make in this regard is the fact that the stability boundaries constitute a set of linear equations. Therefore, the main issue is to handle this set efficiently. In particular, the approach proposed deals with vertices, rather than inequalities, of stabilizing polyhedral sets.

Remark 1: Theorem 3 is a restatement of earlier results available in the literature for the special case of PID controllers, see e.g., research book [7] where the set of all stabilizing PID controllers with fixed proportional gain is shown to be a finite union of convex polygons. In this book, a linear programming approach for the design of PID is also presented, even though linear programming is not strictly necessary since only two design parameters are involved and therefore graphical methods can be used. Subsequent papers along the same direction, and also addressing PID design or lead-lag compensators, are e.g., [9], [10] and [11]. In particular, in [9] a characterization of closed-loop polynomials with a certain even-odd structure is studied by means of the parameter space approach, see e.g., [24]. However, the characterization obtained in [9] is used only for analyzing PID stabilizing controllers and no attempt is made to handle more general classes of controllers. We also recall that a linear programming approach, based on a generalization of the Hermite-Biehler theorem, is developed in [25] for synthesis with fixed structure controllers. This generalization has been subsequently exploited in [26] for interval plants to obtain a stabilizing PID controller through the solution of an integer programming problem but the worst-case complexity seems exponential in the order of the plant. Finally, we recall that the polyhedral characterization of even-odd polynomials has been already observed in [27].

V. Polynomial-Time Algorithms for Stabilization

The proposed approach is divided into two steps. The first step is to compute the so-called marginal stabilizers in polynomial-time. Once this stabilizer is determined, a fixed order controller can be subsequently constructed. This second operation can be also efficiently performed with the aid of one-parameter optimization problem.

A. Computation of Marginal Stabilizers

We first introduce the definition of marginal stabilizer formally.

Definition 2: A marginal stabilizer is a controller $C(s,\theta)$ having the property that the polynomial $p(s,\theta)$ defined in (4) has a fixed number of zeros on the imaginary axis and no zeros in the open right half plane.

Following the discussion in the previous section, the fixed order marginal stabilization problem can be reduced to solving (7). In order to find a marginal stabilizing controller, we construct a square matrix $\bar{\Psi}_i$ which consists of n_θ rows of the matrix Ψ and a vector $\bar{\nu}_i$ which consists of the corresponding n_θ elements of ν. The resulting square linear system is given by

$$\bar{\Psi}_i \theta = \bar{\nu}_i.$$

If $\bar{\Psi}_i$ is invertible (see Lemma 2 below), we can immediately solve the system of linear equations as

$$\theta^{(i)} = \bar{\Psi}_i^{-1} \bar{\nu}_i. \tag{8}$$

This $\theta^{(i)}$ gives a candidate marginal stabilizer. Then, we can check by means of the Routh test if $p(s,\theta)$ has all its zeros in the closed left half plane. If $\theta^{(i)}$ is actually a marginal stabilizer, then we proceed to find a stabilizing controller, as discussed in Section V-B. Otherwise, we repeat this process for another combination of rows of Ψ and corresponding elements of ν. The total number of square matrices $\bar{\Psi}_i$ and vectors $\bar{\nu}_i$ that needs to be computed with this procedure is given by

$$N_{\mathrm{MI}}(n_f, n_\theta) = \frac{n_f!}{n_\theta!(n_f - n_\theta)!}$$

where $n!$ denotes the factorial of n.

Remark 2: For fixed i, $\theta^{(i)}$ can be immediately found by matrix inversion. This requires $O(n_\theta^3)$ operations, see e.g., [28].

In the above procedure, invertibility of the square matrix $\bar{\Psi}_i$ is required. Here we give a technical lemma which ensures the invertibility for all $i = 1, 2, \ldots, N_{\mathrm{MI}}(n_f, n_\theta)$.

Lemma 2: Suppose that $n_f \geq n_\theta$. Then, all square matrices $\bar{\Psi}_i$, $i = 1, 2, \ldots, N_{\mathrm{MI}}(n_f, n_\theta)$, which consist of n_θ rows of the matrix Ψ, are invertible.

Proof: Notice that the matrix Ψ is expressed as

$$\Psi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -\omega_2^2 & \omega_2^4 & \cdots & (-1)^{n_\theta - 1}\omega_2^{2(n_\theta - 1)} \\ 1 & -\omega_3^2 & \omega_3^4 & \cdots & (-1)^{n_\theta - 1}\omega_3^{2(n_\theta - 1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -\omega_{n_f - 1}^2 & \omega_{n_f - 1}^4 & \cdots & (-1)^{n_\theta - 1}\omega_{n_f - 1}^{2(n_\theta - 1)} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

In this matrix, the critical frequencies 0 and ∞ are included as ω_1 and ω_{n_f}, and they correspond to the first row and the last row of Ψ. One can see that the structure of Ψ is similar to that of the Vandermonde matrix, which implies that, if all values ω_i are distinct, the matrix $\bar{\Psi}_i$ is invertible for any i. Indeed, ω_i are all distinct in our case. ■

We now further elaborate on the computational complexity of the problem. In particular, we compare $N_{\mathrm{MI}}(n_f, n_\theta)$ with the number of linear programs $N_{\mathrm{LP}}(n_f)$, which is equal to 2^{n_f}.

Theorem 4: Suppose that $n_f \geq n_\theta$. Then

$$N_{\mathrm{MI}}(n_f, n_\theta) = O(n_f^{n_\theta}). \tag{9}$$

Furthermore, we have

$$N_{\mathrm{MI}}(n_f, n_\theta) \leq N_{\mathrm{LP}}(n_f) \tag{10}$$

where equality is attained only if $n_f = 0$.

Proof: By the properties of the factorial, we have

$$N_{\mathrm{MI}}(n_f, n_\theta) = \frac{n_f(n_f - 1)\cdots(n_f - n_\theta + 1)}{n_\theta!}$$

TABLE I
COMPARISON OF $N_{\rm LP}(n_f)$ AND $N_{\rm MI}(n_f, n_\theta)$

n_f	8	16	32	64
$N_{\rm MI}(n_f, 2)$	28	120	496	2,016
$N_{\rm MI}(n_f, 4)$	70	1,820	35,960	6.3538×10^5
$N_{\rm LP}(n_f)$	256	65,536	4,294,967,296	1.8447×10^{19}

which gives equality (9). The inequality (10) is a direct consequence of a well-known identity

$$\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} = 2^n$$

which is derived from the so-called bimodal theorem. ■

Since we study a fixed order controller problem, we observe that the number of parameters n_θ is fixed. It turns out that, for fixed n_θ, $N_{\rm MI}(n_f, n_\theta)$ is a polynomial function of n_f. Using Lemma 1, we conclude that $N_{\rm MI}(n_f, n_\theta)$ is a polynomial function of n_N, n_D, n_Y, n_Z and n_V. Theorem 4 says that $N_{\rm MI}(n_f, n_\theta)$ is always smaller than $N_{\rm LP}(n_f)$. Some computations of $N_{\rm MI}(n_f, 2)$, $N_{\rm MI}(n_f, 4)$ and $N_{\rm LP}(n_f)$ are given in Table I for different values of n_f. From this table, we conclude that $N_{\rm MI}(n_f, n_\theta)$ is actually much smaller than $N_{\rm LP}(n_f)$.

We now state a result regarding stabilization of the controller parameters. This result easily follows from previous discussions.

Theorem 5: Let $p(s, \theta^{(i)})$ be the polynomial defined in (4) corresponding to $\theta^{(i)}$ computed in (8). There exists a marginal stabilizer if and only if there exists $\theta^{(i)}, i = 1, 2, \ldots, N_{\rm MI}$, such that $p(s, \theta^{(i)})$ has its zeros within the closed left half plane.

Remark 3: The marginally stabilizing controller parameter vector $\theta^{(i)}$, if it exists, is a vertex of a polyhedral set of stabilizing controllers. In this case, the n_θ rows of the corresponding matrix $\bar{\Psi}_i$ and the n_θ elements of $\bar{\nu}_i$ define some of the hyperplanes generating the boundary of a polyhedral set of stabilizing controllers.

B. Computation of a Stabilizing Controller

We now address a subsequent crucial problem: given a marginal stabilizer, determine a fixed order stabilizing controller which places the zeros of the closed-loop polynomial in the open left half plane. To this end, we consider the sensitivity of zeros of $p(s, \theta)$ against perturbations on θ. This approach has been presented for the case of PID controllers in [29], and the method proposed here is an extension to the general case.

Suppose that θ is a marginally stabilizing parameter resulting from (8). Then, $p(s, \theta)$ has n_θ zeros on the imaginary axis and all the other zeros lie in the open left half plane. Suppose that all the imaginary zeros are simple. Let us consider one imaginary zero $j\omega_i$ and study how $j\omega_i$ moves when we perturb θ by $\Delta\theta$. Since $j\omega_i$ is simple, there exists an analytic function $z_i(\Delta\theta)$ in $\|\Delta\theta\| < \epsilon$ for some positive ϵ such that $z_i(0) = j\omega_i$ and $p(z_i(\Delta\theta), \theta + \Delta\theta) = 0$. By differentiating the last equality at $\Delta\theta = 0$, we have

$$\frac{\partial z_i}{\partial \theta_{2k}} = -\left(\left.\frac{\partial p}{\partial s}\right|_{s=j\omega_i} \right)^{-1} \left.\frac{\partial p}{\partial \theta_{2k}}\right|_{s=j\omega_i} \tag{11}$$

for $k = 0, 1, \ldots, n_\theta - 1$. Notice that the quantities on the right-hand side can be easily computed because the inverse of $(\partial p/\partial s)|_{s=j\omega_i}$ is just the reciprocal of a complex number. Since we want $\Delta\theta$ moving all the imaginary zeros inside the left half plane, we consider to solve

$$\begin{bmatrix} {\rm Re}\frac{\partial z_1}{\partial \theta_0} & \cdots & {\rm Re}\frac{\partial z_1}{\partial \theta_{2(n_\theta-1)}} \\ \vdots & \ddots & \vdots \\ {\rm Re}\frac{\partial z_{n_\theta}}{\partial \theta_0} & \cdots & {\rm Re}\frac{\partial z_{n_\theta}}{\partial \theta_{2(n_\theta-1)}} \end{bmatrix} \begin{bmatrix} \Delta\theta_0 \\ \vdots \\ \Delta\theta_{2(n_\theta-1)} \end{bmatrix} = \begin{bmatrix} -1 \\ \vdots \\ -1 \end{bmatrix}. \tag{12}$$

Under the assumption that the matrix on the left-hand side is invertible, we can immediately obtain the desired $\Delta\theta$. After we obtain the desired $\Delta\theta$, we consider a parameter $\theta + \alpha\Delta\theta$ for a positive α. Although a small α gives a stabilizing controller, a large α can be used as well. One recommendable procedure is to use a bisection method for the parameter α.

Remark 4: The above procedure requires polynomial-time operations for n_θ because there is no combinatorial operation involved.

As we have seen, the proposed method may be used under the conditions that $p(s, \theta)$ has simple zeros on the imaginary axis and the matrix on the left-hand side of (12) is invertible. Notice that these conditions are generically satisfied. Otherwise, we may use a randomization based method as an alternative approach. In fact, as a consequence of Theorem 5, the following fact holds true: given a ball (for example ℓ_2) of sufficiently small radius $r > 0$ centered around $\theta^{(i)}$, then there exists a fixed order stabilizer $\theta^{(j)}$ within the ball. Using this observation, we can find $\theta^{(j)}$ using randomization. That is, we generate N points within the ball until we find a stabilizer. This procedure is guaranteed to converge because a stabilizer exists within the ball.

In closing this section, we summarize the proposed algorithm which looks for a stabilizing controller when a randomized parameter $\eta^{(i)}$ is determined according to Algorithm 1 or 2.

Algorithm 3

1. construct Ψ and ν for given $\eta^{(i)}$;
2. for $j := 1, \ldots, N_{\rm MI}(n_f, n_\theta)$ do
   begin
3. compute $\theta^{(j)}$ according to (8);
4. if $\theta^{(j)}$ gives a marginal stabilizer then
   begin
5. compute $\Delta\theta$ according to (12);
6. if a stabilizing parameter $\theta^{(j)} + \alpha\Delta\theta$ is found
   then stop;
   end
   end

Remark 5: From the practical point of view, it is important to deal with more general pole location regions than the left half plane. In various applications we explicitly require that all the closed-loop poles have real part no larger than $-\xi$ where $\xi > 0$. This situation can be easily handled as follows: First, we define

$$\tilde{P}(\tilde{s}) \doteq P(\tilde{s} - \xi)$$

and find (if possible) a marginal stabilizing controller $\tilde{C}(\tilde{s})$ for the plant $\tilde{P}(\tilde{s})$ using the results of this section. Then, a stabilizing controller

$$C(s) \doteq \tilde{C}(s + \xi)$$

can be immediately obtained. We notice that the even-odd structure (2) of the controller is not introduced into the the original $C(s)$, but into the shifted controller $\tilde{C}(\tilde{s})$. We also remark that with this method the perturbation argument previously discussed may be avoided.

VI. Convexity Characterization for H_∞ Performance

In this section, we study H_∞ performance of the sensitivity and complementary sensitivity functions defined as

$$S(s) \doteq \frac{1}{1 + P(s)C(s)};$$
$$T(s) \doteq \frac{P(s)C(s)}{1 + P(s)C(s)}.$$

In particular, we consider the following H_∞ performance problem: given a weighting (stable) transfer function

$$W(s) \doteq \frac{N_W(s)}{D_W(s)}$$

find a fixed structure stabilizing controller which satisfies the constraint $\|W(s)S(s)\|_\infty < 1$. In this and in the next section, we focus on this sensitivity problem, but we observe that a performance problem involving the complementary sensitivity function can be solved with similar techniques.

Note that $S(\infty) = 1$ since $P(s)$ is strictly proper and $C(s)$ is proper. Therefore, $|W(\infty)| < 1$ is a necessary condition for the desired controller to exist. Henceforth, we require the weighting transfer function $W(s)$ to satisfy this condition.

First, we rewrite the weighted sensitivity function as

$$\begin{aligned} W(s)S(s) &= \frac{W(s)}{1+P(s)C(s)} \\ &= \frac{N_W(s)D_P(s)D_C(s)}{D_W(s)(N_P(s)N_C(s)+D_P(s)D_C(s))} \\ &= \frac{N_W(s)D_P(s)(Z(s^2)+sV(s^2))}{D_W(s)[N_P(s)(X(s^2)+sY(s^2))+D_P(s)(Z(s^2)+sV(s^2))]}. \end{aligned}$$

Next, we state a technical lemma, which is an extension of a discussion in [30, p. 6].

Lemma 3: Let us consider a proper transfer function $G(s)$ of the form

$$G(s) = \frac{N_G(s)}{D_G(s)}$$

where $N_G(s)$ and $D_G(s)$ are numerator and denominator polynomials, respectively. Suppose that $|G(\infty)| < 1$. Then, $G(s)$ is stable and satisfies $\|G(s)\|_\infty < 1$ if and only if the complex coefficient polynomial $N_G(s) + e^{j\phi}D_G(s)$ has no zeros in the closed right half plane for all $\phi \in [0, 2\pi)$.

Proof: Suppose that $G(s)$ is stable and satisfies $\|G(s)\|_\infty < 1$. Then, $N_G(s)/D_G(s) = -e^{j\phi}$ cannot hold for any s in the closed right half plane and for any $\phi \in [0, 2\pi)$. This proves the "only if" part. To prove the "if" part, suppose that $G(s)$ is not stable. Then, it has a pole in the closed right half plane. Since $|G(\infty)| < 1$, continuity of $G(s)$ implies that $N_G(s)/D_G(s) = -e^{j\phi}$ holds for some s in the closed right half plane and for some $\phi \in [0, 2\pi)$. Similar reasoning is possible also in the case of $\|G(s)\|_\infty \geq 1$. ■

Since $|W(\infty)S(\infty)| < 1$ by assumption, the H_∞ performance problem can be restated as follows: find a fixed structure controller such that the parameterized polynomial

$$\begin{aligned} p(s,\phi) = N_W(s)D_P(s)(Z(s^2) + sV(s^2)) \\ + e^{j\phi}D_W(s)[N_P(s)(X(s^2) \\ + sY(s^2)) + D_P(s)(Z(s^2) + sV(s^2))] \end{aligned}$$

has no zeros in the closed right half plane for all $\phi \in [0, 2\pi)$. Since we are interested in the case when the coefficients of $X(s^2)$ represent the deterministic parameters and the coefficients of $Y(s^2), Z(s^2)$ and $V(s^2)$ are selected using the randomized algorithms given in Section III, we need to write explicitly the dependence on $X(s^2)$. To this end, letting

$$\begin{aligned} p_0(s,\phi) &= N_W(s)D_P(s)(Z(s^2)+sV(s^2)) \\ &\quad + e^{j\phi}D_W(s)[sN_P(s)Y(s^2)+D_P(s)(Z(s^2)+sV(s^2))] \\ p_1(s,\phi) &= e^{j\phi}D_W(s)N_P(s) \end{aligned}$$

we obtain

$$p(s,\phi) = p_0(s,\phi) + p_1(s,\phi)X(s^2).$$

Equivalently, to exploit the parameters θ of the controller $C(s,\theta)$, we write

$$\begin{aligned} p(s,\phi,\theta) = p_0(s,\phi) + p_1(s,\phi)[\theta_0 + \theta_2 s^2 + \cdots \\ + \theta_{2(n_\theta-1)}s^{2(n_\theta-1)}]. \end{aligned} \quad (13)$$

Clearly, this is a stabilization problem similar to that previously studied in Sections IV and V, even though we now need to deal with the additional parameter ϕ. Henceforth, we define the even/odd polynomials

$$\begin{aligned} R_{00}(s^2) &\doteq \text{even part of } \{N_W(s)D_P(s)(Z(s^2) + sV(s^2))\} \\ I_{00}(s^2) &\doteq \frac{1}{s}\text{ odd part of } \{N_W(s)D_P(s)(Z(s^2) + sV(s^2))\} \\ R_{01}(s^2) &\doteq \text{even part of } \{D_W(s)[sN_P(s)Y(s^2) \\ &\qquad + D_P(s)(Z(s^2) + sV(s^2))]\} \\ I_{01}(s^2) &\doteq \frac{1}{s}\text{ odd part of } \{D_W(s)[sN_P(s)Y(s^2) \\ &\qquad + D_P(s)(Z(s^2) + sV(s^2))]\} \\ R_{11}(s^2) &\doteq \text{even part of } \{D_W(s)N_P(s)\} \\ I_{11}(s^2) &\doteq \frac{1}{s}\text{ odd part of } \{D_W(s)N_P(s)\}. \end{aligned}$$

Taking $s = j\omega$, we write

$$\begin{aligned} p_0(j\omega,\phi) &= R_{00}(-\omega^2) + j\omega I_{00}(-\omega^2) \\ &\quad + e^{j\phi}(R_{01}(-\omega^2) + j\omega I_{01}(-\omega^2)) \\ p_1(j\omega,\phi) &= e^{j\phi}(R_{11}(-\omega^2) + j\omega I_{11}(-\omega^2)). \end{aligned} \tag{14}$$

We now state a lemma which provides the critical frequencies. Here, a critical frequency is a value of ω that achieves $p(j\omega,\phi,\theta) = 0$ for some θ when ϕ is fixed. This result is an extension of (6) previously obtained for stabilization.

Lemma 4: For fixed $\phi \in [0, 2\pi)$, the critical frequencies are given by either ∞ or a solution of the polynomial equation

$$f_0(\omega) + \sin\phi f_s(\omega) + \cos\phi f_c(\omega) = 0 \tag{15}$$

where

$$\begin{aligned} f_0(\omega) &= \omega I_{01}(-\omega^2)R_{11}(-\omega^2) - \omega R_{01}(-\omega^2)I_{11}(-\omega^2) \\ f_s(\omega) &= -\omega^2 I_{00}(-\omega^2)I_{11}(-\omega^2) - R_{00}(-\omega^2)R_{11}(-\omega^2) \\ f_c(\omega) &= \omega I_{00}(-\omega^2)R_{11}(-\omega^2) - \omega R_{00}(-\omega^2)I_{11}(-\omega^2). \end{aligned}$$

Proof: By reasoning similar to Section IV, a critical frequency for a fixed $\phi \in [0, 2\pi)$ is either infinity or a solution of

$$\mathrm{Re}\, p_0(j\omega,\phi)\mathrm{Im}\, p_1(j\omega,\phi) - \mathrm{Im} p_0(j\omega,\phi)\mathrm{Re}\, p_1(j\omega,\phi) = 0.$$

There is no need to specialize $\omega = 0$ this time because the imaginary parts of $p_0(j\omega,\phi)$ and $p_1(j\omega,\phi)$ are not always equal to zero at $\omega = 0$. For convenience of computation, we consider the following equation equivalent to the above:

$$\begin{aligned} &\mathrm{Re}[e^{-j\phi}p_0(j\omega,\phi)]\mathrm{Im}[e^{-j\phi}p_1(j\omega,\phi)] \\ &\quad - \mathrm{Im}[e^{-j\phi}p_0(j\omega,\phi)]\mathrm{Re}[e^{-j\phi}p_1(j\omega,\phi)] = 0. \end{aligned}$$

Substitution of the explicit expressions

$$\begin{aligned} \mathrm{Re}[e^{-j\phi}p_0(j\omega,\phi)] &= R_{01}(-\omega^2) \\ &\quad + \omega I_{00}(-\omega^2)\sin\phi + R_{00}(-\omega^2)\cos\phi \\ \mathrm{Im}[e^{-j\phi}p_0(j\omega,\phi)] &= \omega I_{01}(-\omega^2) \\ &\quad - R_{00}(-\omega^2)\sin\phi + \omega I_{00}(-\omega^2)\cos\phi \\ \mathrm{Re}[e^{-j\phi}p_1(j\omega,\phi)] &= R_{11}(-\omega^2) \\ \mathrm{Im}[e^{-j\phi}p_1(j\omega,\phi)] &= \omega I_{11}(-\omega^2) \end{aligned}$$

gives the statement of the lemma. ■

Note that the critical frequencies $\Omega(\phi) = \{\omega_1(\phi), \omega_2(\phi), \ldots, \omega_{n_f}(\phi)\}$ are now parameterized with ϕ, where its cardinality n_f can depend on ϕ. Then, the same procedure employed for stabilization gives the following characterization result, which is an extension of Theorem 3. It also generalizes previous results obtained in [11], [18], [19] for the special case of PID and lead-lag controllers. Here, we let Γ be the set of all deterministic parameters θ that give an H_∞ controller $C(s,\theta)$ satisfying $\|W(s)S(s)\|_\infty \leq 1$.

Theorem 6: Suppose that the randomized parameter vector η is selected. Then, the set Γ, defined above, is either empty or is given by

$$\Gamma = \bigcap_{\phi\in[0,2\pi)} \Gamma(\phi)$$

where $\Gamma(\phi)$ is a union of a finite number of polyhedral sets for fixed ϕ.

Proof: Fix $\phi \in [0, 2\pi)$ and write $\Gamma(\phi)$ to describe the set of all θ's for which the equation $p(s,\phi,\theta) = 0$ has its roots in the closed left half plane. For each critical frequency $\omega_i(\phi) \in \Omega(\phi)$, the equation $p(j\omega_i(\phi),\phi,\theta) = 0$ is affine in θ and can be written as

$$\psi^T(\omega_i(\phi))\theta = \nu(\omega_i(\phi),\phi)$$

where[2]

$$\begin{aligned} \psi^T(\omega_i(\phi)) &= \left[\, 1\ -\omega_i^2(\phi)\ \omega_i^4(\phi)\ \cdots\ (-1)^{n_\theta-1}\omega_i^{2(n_\theta-1)}(\phi) \,\right] \\ \nu(\omega_i(\phi),\phi) &= -\frac{p_0(j\omega_i(\phi),\phi)}{p_1(j\omega_i(\phi),\phi)}. \end{aligned}$$

This stands for a hyperplane in the space of θ. The hyperplanes defined in this way divide the space of θ into a finite number of polyhedral sets. By construction, if some θ in the interior of a polyhedral set belongs to $\Gamma(\phi)$, this entire polyhedral set is included in $\Gamma(\phi)$. This means that $\Gamma(\phi)$ is a union of a finite number of polyhedral sets provided it is not empty. Since the desired θ belongs to $\Gamma(\phi)$ for all $\phi \in [0, 2\pi)$, the statement of the theorem follows. ■

In closing this section, we remark that, if ω is a critical frequency for ϕ, $-\omega$ is also a critical frequency for $2\pi - \phi$. This is a direct consequence of (15) given in Lemma 4. For these ω and ϕ, we see that

$$\begin{aligned} &\frac{p_0(j\omega,\phi)}{p_1(j\omega,\phi)} \\ &\quad = \frac{\mathrm{Re}[e^{-j\phi}p_0(j\omega,\phi)]}{\mathrm{Re}[e^{-j\phi}p_1(j\omega,\phi)]} \\ &\quad = \frac{R_{01}(-\omega^2) + \omega I_{00}(-\omega^2)\sin\phi + R_{00}(-\omega^2)\cos\phi}{R_{11}(-\omega^2)} \\ &\quad = \frac{p_0(-j\omega, 2\pi-\phi)}{p_1(-j\omega, 2\pi-\phi)}. \end{aligned}$$

That is, the hyperplane of ω for ϕ is identical to that of $-\omega$ for $2\pi - \phi$. We therefore see that only the *nonnegative* critical frequencies should be considered for H_∞ performance as well as for stabilization.

VII. Efficient Algorithms for H_∞ Performance

We construct an algorithm for the H_∞ performance problem based on the result of the previous section. In particular, we take n_ϕ grid points $\phi_1, \phi_2, \ldots, \phi_{n_\phi}$ and approximate the set Γ by

$$\tilde{\Gamma} = \bigcap_{i=1}^{n_\phi} \Gamma(\phi_i).$$

[2]The definitions of ψ and ν should be appropriately modified when $\omega_i(\phi) = \infty$; see details in Section IV.

As we have seen, if ϕ_i and $\phi_j = 2\pi - \phi_i$ are both included in the grid points, the negative critical frequencies can be omitted in the following. In order to study the properties of $\tilde{\Gamma}$, we construct the matrix Ψ and the vector ν as follows

$$\Psi = \begin{bmatrix} \psi^T(\omega_1(\phi_1)) \\ \vdots \\ \psi^T(\omega_{n_f}(\phi_1)) \\ \psi^T(\omega_1(\phi_2)) \\ \vdots \\ \psi^T(\omega_{n_f}(\phi_2)) \\ \vdots \\ \psi^T(\omega_1(\phi_{n_\phi})) \\ \vdots \\ \psi^T(\omega_{n_f}(\phi_{n_\phi})) \end{bmatrix}, \quad \nu = \begin{bmatrix} \nu(\omega_1(\phi_1), \phi_1) \\ \vdots \\ \nu(\omega_{n_f}(\phi_1), \phi_1) \\ \nu(\omega_1(\phi_2), \phi_2) \\ \vdots \\ \nu(\omega_{n_f}(\phi_2), \phi_2) \\ \vdots \\ \nu(\omega_1(\phi_{n_\phi}), \phi_{n_\phi}) \\ \vdots \\ \nu(\omega_{n_f}(\phi_{n_\phi}), \phi_{n_\phi}) \end{bmatrix}.$$

The matrix Ψ has $\sum_{i=1}^{n_\phi} n_f(\phi_i)$ rows and so does the vector ν. The number of the columns of Ψ is n_θ.

Each row of Ψ and ν expresses a hyperplane in the space of θ. By the Proof of Theorem 6, these hyperplanes construct the boundaries of $\tilde{\Gamma}$. Moreover, each vertex of $\tilde{\Gamma}$ is a solution of $\bar{\Psi}\theta = \bar{\nu}$, where $\bar{\Psi}$ is an invertible matrix constructed by appropriate n_θ rows of Ψ and $\bar{\nu}$ is the corresponding subvector of ν. Now, let us compute these vertices. We consider

$$\theta^{(i)} = \bar{\Psi}_i^{-1}\bar{\nu}_i \tag{16}$$

for all $n_\theta \times n_\theta$ submatrices $\bar{\Psi}_i$ and the corresponding subvectors $\bar{\nu}_i$. Here, the index i runs from 1 to

$$N_{\mathrm{MI}} = \frac{\left(\sum_{j=1}^{n_\phi} n_f(\phi_j)\right)!}{n_\theta! \left[\left(\sum_{j=1}^{n_\phi} n_f(\phi_j)\right) - n_\theta\right]!}.$$

When $\bar{\Psi}_i$ is not invertible, $\theta^{(i)}$ is not defined. Some of the computed $\theta^{(i)}$ are vertices of $\tilde{\Gamma}$ while others are not. The vector $\theta^{(i)}$ is a vertex of $\tilde{\Gamma}$ if and only if the equation $p(s, \phi_k, \theta^{(i)}) = 0$ has its roots only in the closed left half plane for all $k = 1, 2, \ldots, n_\phi$. Hence, we can find a vertex of $\tilde{\Gamma}$ using the Routh test.

Note in the expression of N_{MI} that n_ϕ is fixed and independent of n_θ. Hence, the same complexity results derived in Section V-A can be established here. In particular, n_ϕ may be computed using the so-called pancake formula, as in [11].

Once a vertex of $\tilde{\Gamma}$ is determined, the same approach outlined in Section V-B can be used to compute a stabilizing controller. In particular, we could follow either the sensitivity approach given in (12) or the randomization method proposed at the end of the section. Application of the sensitivity approach needs some care. In the stabilization case, a small enough α always provides a stabilizer, which implies that an appropriate α can be searched via bisection. This is not true in the present case. Indeed, a too small α may not give a stabilizer because we start from a vertex of $\tilde{\Gamma}$, which is a superset of Γ. Hence, we need gridding in order to find an appropriate value of α.

Algorithm 4

---

1. set n_ϕ, Δn_ϕ, and n_ϕ^u;
2. while $n_\phi \leq n_\phi^u$ do
   begin
3. construct Ψ and ν for given $\eta^{(i)}$;
4. for $j := 1, \ldots, N_{\mathrm{MI}}$ do
   begin
5. compute $\theta^{(j)}$ according to (16);
6. if $\theta^{(j)}$ is a vertex of $\tilde{\Gamma}$ then
   begin
7. compute $\Delta\theta$ according to (12);
8. if the parameter $\theta^{(j)} + \alpha\Delta\theta$ fulfills
   $\|W(s)S(s)\|_\infty < 1$ then stop;
   end
   end
9. $n_\phi := n_\phi + \Delta n_\phi$;
   end

Remark 6: We can handle multi-objective cases in a similar way. Let us consider an example: Find a controller which satisfies $\|W_1(s)S(s)\|_\infty < 1$ and $\|W_2(s)T(s)\|_\infty < 1$. We first compute (Ψ_1, ν_1) and (Ψ_2, ν_2) each of which corresponds to a set of hyperplanes related to each specification, and construct

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}, \qquad \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}.$$

Then, we perform the proposed algorithm for (Ψ, ν). In this case, we see that we are now looking for a controller which satisfies both specifications.

VIII. Further Extensions: Stabilization of Interval Plants

In this section, we study extensions of the stabilization results previously given to the case when the fixed plant $P(s)$ defined in (1) is replaced with a single-input single-output strictly proper interval plant of the form

$$P(s, q, r) = \frac{N_P(s, q)}{D_P(s, r)} \tag{17}$$

where $N_P(s, q)$ and $D_P(s, r)$ are interval polynomials of order n_N and n_D, respectively. That is, the coefficients of these polynomials vary between given upper and lower bounds

$$N_P(s, q) = q_0 + q_1 s + q_2 s^2 + \cdots + q_{n_N} s^{n_N} \tag{18}$$

$$D_P(s, r) = r_0 + r_1 s + r_2 s^2 + \cdots + r_{n_D} s^{n_D} \tag{19}$$

where $q_i \in [q_i^-, q_i^+]$, $i = 0, 1, \ldots, n_N$ and $r_i \in [r_i^-, r_i^+]$, $i = 0, 1, \ldots, n_D$. The closed-loop polynomial $p(s,q,r)$ is given by

$$p(s,q,r) = N_P(s,q)N_C(s) + D_P(s,r)D_C(s)$$

where the degree of $p(s,q,r)$ is assumed to be fixed for all possible q_i and r_i within the assigned bounds.

Associated to the interval polynomial $N_P(s,q)$ we consider the Kharitonov polynomials

$$\begin{aligned}
N_1(s) &\doteq q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + \cdots \\
N_2(s) &\doteq q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \cdots \\
N_3(s) &\doteq q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + \cdots \\
N_4(s) &\doteq q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + \cdots. \quad (20)
\end{aligned}$$

Similarly, we define the Kharitonov polynomials associated to $D_P(s,r)$

$$\begin{aligned}
D_1(s) &\doteq r_0^+ + r_1^+ s + r_2^- s^2 + r_3^- s^3 + r_4^+ s^4 + r_5^+ s^5 + \cdots \\
D_2(s) &\doteq r_0^- + r_1^- s + r_2^+ s^2 + r_3^+ s^3 + r_4^- s^4 + r_5^- s^5 + \cdots \\
D_3(s) &\doteq r_0^- + r_1^+ s + r_2^+ s^2 + r_3^- s^3 + r_4^- s^4 + r_5^+ s^5 + \cdots \\
D_4(s) &\doteq r_0^+ + r_1^- s + r_2^- s^2 + r_3^+ s^3 + r_4^+ s^4 + r_5^- s^5 + \cdots. \quad (21)
\end{aligned}$$

We consider a fixed order controller parameterized as in (2) and study closed-loop stabilization. As in previous sections, without loss of generality, we study the case when the deterministic parameters are the coefficients of $X(s^2)$ and we assume that the randomized parameters, i.e., the coefficients of $Y(s^2)$, $Z(s^2)$ and $V(s^2)$, are selected using the randomized algorithms given in Section III. We now state a result given in [31] regarding stability of interval plants with fixed order controllers, see also [30] for a different proof.

Lemma 5: Consider an interval plant $P(s,q,r)$ of the form (17) with associated Kharitonov polynomials (20) and (21) for the numerator and denominator, respectively. For fixed $k, \ell \in \{1,2,3,4\}$ and $\lambda \in [0,1]$, define

$$\begin{aligned}
N_{k,\ell}(s,\lambda) &\doteq \lambda N_k(s) + (1-\lambda)N_\ell(s) \\
D_{k,\ell}(s,\lambda) &\doteq \lambda D_k(s) + (1-\lambda)D_\ell(s).
\end{aligned}$$

Then, the closed-loop polynomial $p(s,q,r)$ is stable for all $q_i \in [q_i^-, q_i^+]$, $i = 0, 1, \ldots, n_N$ and $r_i \in [r_i^-, r_i^+]$, $i = 0, 1, \ldots, n_D$, if and only if either one of the following conditions is satisfied:

1) The parameterized polynomials

$$\begin{aligned}
p(s,\lambda,i_1,i_2,i_3) = N_{i_1}(s)(X(s^2) + sY(s^2)) \\
+ D_{i_2,i_3}(s,\lambda)(Z(s^2) + sV(s^2)) \quad (22)
\end{aligned}$$

with $i_1 \in \{1,2,3,4\}$ and $(i_2,i_3) \in \{(1,3),(1,4),(2,3),(2,4)\}$ are stable for all $\lambda \in [0,1]$;

2) The parameterized polynomials

$$\begin{aligned}
p(s,\lambda,i_1,i_2,i_3) = N_{i_1,i_2}(s,\lambda)(X(s^2) + sY(s^2)) \\
+ D_{i_3}(s)(Z(s^2) + sV(s^2)) \quad (23)
\end{aligned}$$

with $(i_1,i_2) \in \{(1,3),(1,4),(2,3),(2,4)\}$ and $i_3 \in \{1,2,3,4\}$ are stable for all $\lambda \in [0,1]$.

We now develop a method which is an extension of the techniques used for stabilization and performance of a fixed plant. Since we are interested in the case when the coefficients of $X(s^2)$ represent the deterministic parameters, we exploit the dependence on this polynomial. In particular, for the first condition in the lemma above, letting

$$\begin{aligned}
p_0(s,i_2,i_3) &= (D_{i_2}(s) - D_{i_3}(s))(Z(s^2) + sV(s^2)) \\
p_1(s,i_1,i_3) &= sN_{i_1}(s)Y(s^2) + D_{i_3}(s)(Z(s^2) + sV(s^2)) \\
p_2(s,i_1) &= N_{i_1}(s)
\end{aligned}$$

where $i_1 \in \{1,2,3,4\}$, $(i_2,i_3) \in \{(1,3),(1,4),(2,3),(2,4)\}$, we write (22) as

$$p(s,\lambda,i_1,i_2,i_3) = \lambda p_0(s,i_2,i_3) + p_1(s,i_1,i_3) + p_2(s,i_1)X(s^2).$$

For the second condition of the lemma, defining

$$\begin{aligned}
p_0(s,i_1,i_2) &= sY(s^2)(N_{i_1}(s) - N_{i_2}(s)) \\
p_1(s,i_2,i_3) &= sN_{i_2}(s)Y(s^2) + D_{i_3}(s)(Z(s^2) + sV(s^2)) \\
p_2(s,i_1,i_2) &= N_{i_1}(s) - N_{i_2}(s) \\
p_3(s,i_2) &= N_{i_2}(s)
\end{aligned}$$

where $(i_1,i_2) \in \{(1,3),(1,4),(2,3),(2,4)\}$, $i_3 \in \{1,2,3,4\}$, we write (23) as

$$\begin{aligned}
p(s,\lambda,i_1,i_2,i_3) = \lambda p_0(s,i_1,i_2) + p_1(s,i_2,i_3) \\
+ (\lambda p_2(s,i_1,i_2) + p_3(s,i_2))X(s^2).
\end{aligned}$$

The next two results provide the critical frequencies corresponding to the two conditions (22) and (23) of Lemma 5. We observe that the parameter λ enters linearly and quadratically into the conditions given in Lemmas 6 and 7, respectively. These results are stated in terms of real and imaginary parts of the polynomials p_0, p_1, p_2 and p_3. Clearly, these real and imaginary parts can be also written in terms of problem data, i.e., in terms of plant and controller polynomials, by means of lengthy but straightforward computations.

Lemma 6: For fixed ω, we define real and imaginary parts of the polynomials p_0, p_1 and p_2

$$\begin{aligned}
p_0(j\omega,i_2,i_3) &= R_0(-\omega^2,i_2,i_3) + j\omega I_0(-\omega^2,i_2,i_3) \\
p_1(j\omega,i_1,i_3) &= R_1(-\omega^2,i_1,i_3) + j\omega I_1(-\omega^2,i_1,i_3) \\
p_2(j\omega,i_1) &= R_2(-\omega^2,i_1) + j\omega I_2(-\omega^2,i_1)
\end{aligned}$$

where $i_1 \in \{1,2,3,4\}$, $(i_2,i_3) \in \{(1,3),(1,4),(2,3),(2,4)\}$. Then, for fixed $\lambda \in [0,1]$, the critical frequencies corresponding to the polynomial (22) are given by either 0, ∞, or a solution of the polynomial equation

$$\lambda f_1(\omega^2,i_1,i_2,i_3) + f_0(\omega^2,i_1,i_3) = 0 \quad (24)$$

where

$$f_1(\omega^2, i_1, i_2, i_3) = R_0(-\omega^2, i_2, i_3)I_2(-\omega^2, i_1) - I_0(-\omega^2, i_2, i_3)R_2(-\omega^2, i_1)$$
$$f_0(\omega^2, i_1, i_3) = R_1(-\omega^2, i_1, i_3)I_2(-\omega^2, i_1) - I_1(-\omega^2, i_1, i_3)R_2(-\omega^2, i_1).$$

Proof: Using the definitions of p_0, p_1 and p_2, the critical frequencies are given by the solutions of the equation

$$(\lambda R_0(-\omega^2, i_2, i_3) + R_1(-\omega^2, i_1, i_3))I_2(-\omega^2, i_1) - (\lambda I_0(-\omega^2, i_2, i_3) + I_1(-\omega^2, i_1, i_3))R_2(-\omega^2, i_1) = 0.$$

The proof is concluded simply using the definitions of $f_1(\omega^2, i_1, i_2, i_3)$ and $f_0(\omega^2, i_1, i_3)$. ■

Lemma 7: For fixed ω, we define real and imaginary parts of the polynomials p_0, p_1, p_2 and p_3

$$p_0(j\omega, i_1, i_2) = R_0(-\omega^2, i_1, i_2) + j\omega I_0(-\omega^2, i_1, i_2)$$
$$p_1(j\omega, i_2, i_3) = R_1(-\omega^2, i_2, i_3) + j\omega I_1(-\omega^2, i_2, i_3)$$
$$p_2(j\omega, i_1, i_2) = R_2(-\omega^2, i_1, i_2) + j\omega I_2(-\omega^2, i_1, i_2)$$
$$p_3(j\omega, i_2) = R_3(-\omega^2, i_2) + j\omega I_3(-\omega^2, i_2)$$

where $(i_1, i_2) \in \{(1,3), (1,4), (2,3), (2,4)\}$, $i_3 \in \{1,2,3,4\}$. Then, for fixed $\lambda \in [0,1]$, the critical frequencies corresponding to the polynomial (23) are given by either 0, ∞, or a solution of the polynomial equation

$$\lambda^2 f_2(\omega^2, i_1, i_2) + \lambda f_1(\omega^2, i_1, i_2, i_3) + f_0(\omega^2, i_2, i_3) = 0 \quad (25)$$

where

$$f_2(\omega^2, i_1, i_2) = R_0(-\omega^2, i_1, i_2)I_2(-\omega^2, i_1, i_2) - I_0(-\omega^2, i_1, i_2)R_2(-\omega^2, i_1, i_2)$$
$$f_1(\omega^2, i_1, i_2, i_3) = R_0(-\omega^2, i_1, i_2)I_3(-\omega^2, i_2) + R_1(-\omega^2, i_2, i_3)I_2(-\omega^2, i_1, i_2) - I_0(-\omega^2, i_1, i_2)R_3(-\omega^2, i_2) - I_1(-\omega^2, i_2, i_3)R_2(-\omega^2, i_1, i_2)$$
$$f_0(\omega^2, i_2, i_3) = R_1(-\omega^2, i_2, i_3)I_3(-\omega^2, i_2) - I_1(-\omega^2, i_2, i_3)R_3(-\omega^2, i_2).$$

The proof of this result follows along the same line of proof of Lemma 6 and it is not reported here.

We now have a stabilization problem similar to that previously studied in Sections IV and V. However, the additional parameter λ and the indices i_1, i_2 and i_3 should be taken into account. In particular, for the parameter λ we generate n_λ grid points in the interval $[0,1]$. Then, for fixed i_1, i_2, i_3, the critical frequencies can be computed using Lemma 6 or 7 obtaining the set

$$\Omega(\lambda, i_1, i_2, i_3) = \{\omega_1(\lambda, i_1, i_2, i_3), \omega_2(\lambda, i_1, i_2, i_3), \ldots, \omega_{n_f}(\lambda, i_1, i_2, i_3)\}.$$

The hyperplanes defining the stability boundary are of the form

$$\psi^T(\omega_i(\lambda_k, i_1, i_2, i_3))\theta = \nu(\omega_i(\lambda_k, i_1, i_2, i_3), \lambda_k, i_1, i_2, i_3)$$

where $\omega_i(\lambda_k, i_1, i_2, i_3) \in \Omega(\lambda_k, i_1, i_2, i_3)$, $k = 1, 2, \ldots, n_\lambda$. Subsequently, two linear systems, one for each condition stated in Lemma 5, can be constructed. However, for the first case, the matrix Ψ and the vector ν also depend on the indices i_1, i_2, i_3. Considering the 16 combinations of these indices, we have Ψ and ν. A similar argument can be made for the second condition of Lemma 5. The matrix inversion algorithm similar to (16) can be used. Subsequently, a marginal stabilizer can be easily obtained with the procedure explained in Section VII. For simplicity, these details are not reported here.

IX. Application Examples

A. Stabilization

In this first example, we illustrate the main idea of this paper, where a stabilization problem is considered for simplicity.

Let us consider a single-input single-output plant

$$P(s) = \frac{N_P(s)}{D_P(s)} = \frac{17(1+s)(1+16s)(1-s+s^2)}{s(1-s)(90-s)(1+s+4s^2)} \quad (26)$$

and a fixed order controller $C(s)$ of the form

$$C(s) = \frac{\theta_0 + \alpha_0 s + \theta_2 s^2}{\beta_0 + \mu_0 s + \beta_2 s^2}. \quad (27)$$

We follow the first classification of the deterministic and randomized parameters in Section II, i.e.

$$\theta = [\,\theta_0 \quad \theta_2\,]^T, \eta = [\,\alpha_0 \quad \beta_0 \quad \beta_2 \quad \mu_0\,]^T.$$

We set the parameters in Theorem 1 as $\epsilon = 0.01$ and $\delta = 0.01$, which implies $N_1 = 459$ in Algorithm 1. That is, if we can not find a stabilizing controller within 459 tries, we conclude that this stabilization problem is infeasible in the sense of Theorem 1, where the measure $\mathcal{P}(\mathcal{A})$ can be evaluated with Algorithm 2 with $N_2 = 26\,492$. For the controller parameters, we set $\beta_0 = 1$ without loss of generality, and randomly generated $\alpha_0 \in [-5, 5]$, $\beta_2 \in [-5, 5]$, and $\mu_0 \in [-5, 5]$ according to uniform distributions within these intervals.

We run Algorithm 1, together with Algorithm 3, for the search of deterministic parameters θ, obtaining a stabilizing controller for $i = 16$. The details are as follows.

The randomized parameters were selected as

$$\eta^{(16)} = [\,-0.7568 \quad 1 \quad -0.8570 \quad -0.3136\,]^T.$$

For this $\eta^{(16)}$, we obtained five critical frequencies

$$\Omega = \{\, 0,\ 9.9271,\ 1.0514,\ 0.6907,\ 0.1428 \,\}.$$

Since we have two deterministic parameters, a candidate marginal stabilizer can be simply computed with two of these critical frequencies. For $j = 8$, i.e., when we chose 1.0514 and 0.6907, we had

$$\bar{\Psi}_8 = \begin{bmatrix} 1 & -1.1055 \\ 1 & -0.4771 \end{bmatrix}, \bar{\nu}_8 = \begin{bmatrix} 2.0620 \\ 0.3310 \end{bmatrix}$$

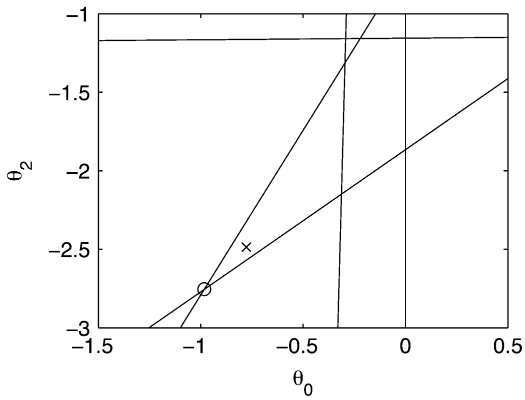

Fig. 1. Deterministic parameter space of the stabilizing controller.

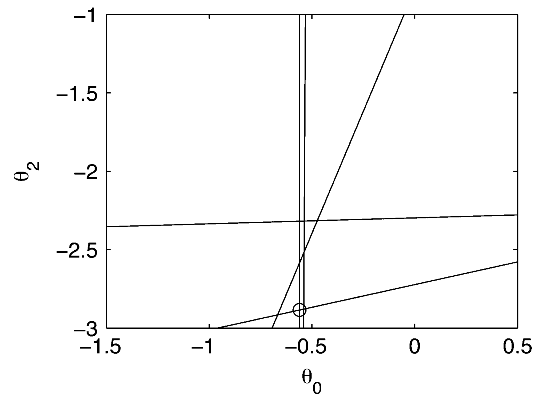

Fig. 2. Deterministic parameter space of the ξ-stabilizing controller.

and obtained

$$\theta^{(8)} = [-0.9831 \quad -2.7546]^T$$

by solving the corresponding linear equation. Actually, with this $\theta^{(8)}$, we have a marginally stabilizing controller

$$C_m(s) = \frac{-0.9831 - 0.7568s - 2.7546s^2}{1 - 0.3136s - 0.8570s^2}$$

and the poles of the corresponding closed-loop system are

$$-127.36, \quad -0.7258, \quad -0.1000, \quad \pm 1.0514j, \quad \pm 0.6907j$$

which actually shows that there exist two pairs of marginal poles in this closed-loop system.

Then, we computed the desired direction of the perturbation

$$\Delta\theta = [4.1183 \quad 5.3715]^T.$$

With the step parameter $\alpha = 0.05$, we finally obtained a stabilizing parameter $\theta_s = \theta^{(8)} + \alpha\Delta\theta$

$$\theta_s = [-0.7772 \quad -2.4861]^T$$

and thus we see that a stabilizing controller can be described as

$$C_s(s) = \frac{-0.7772 - 0.7568s - 2.4861s^2}{1 - 0.3136s - 0.8570s^2}. \qquad (28)$$

In fact, the poles of the corresponding closed-loop system are

$$-105.90, \quad -0.5934, \quad -0.1244, \quad -0.0601 \pm 1.0544j, \quad -0.0705 \pm 0.6612j$$

which shows that all of them have negative real parts. Fig. 1 shows the deterministic parameter space, where five lines denote the candidate stability boundaries which correspond to three critical frequencies, and "o" and "x" denote $\theta^{(8)}$ and θ_s, respectively.

Since some of the closed-loop poles are close to the imaginary axis, we employed the pole shifting technique explained in Remark 5. We set $\xi = 0.2$ and redefined the plant as $\tilde{P}(\tilde{s}) = P(\tilde{s} - \xi)$. Then, we executed Algorithms 1 and 3 obtaining a stabilizing controller for $i = 35$. The details are now given.

The randomized parameters were chosen as

$$\eta^{(35)} = [0.1551 \quad 1 \quad -1.6605 \quad -0.6709]^T.$$

For this $\eta^{(35)}$, we obtained five critical frequencies

$$\Omega = \{\, 0,\ 5.1513,\ 1.8616,\ 0.5665,\ 0.0640 \,\}.$$

For $j = 2$, i.e., when we chose 0 and 1.8616, we had

$$\bar{\Psi}_2 = \begin{bmatrix} 1 & 0 \\ 1 & -3.4656 \end{bmatrix}, \bar{\nu}_2 = \begin{bmatrix} -0.5602 \\ 9.4350 \end{bmatrix}$$

and obtained

$$\theta^{(2)} = [-0.5602 \quad -2.8841]^T$$

by solving the corresponding linear equation. That is, with this $\theta^{(2)}$, we have a marginally stabilizing controller for the modified plant $\tilde{P}(\tilde{s})$ as

$$\tilde{C}_s(\tilde{s}) = \frac{-0.5602 + 0.1551\tilde{s} - 2.8841\tilde{s}^2}{1 - 0.6709\tilde{s} - 1.6605\tilde{s}^2}.$$

We therefore obtained a stabilizing controller for the original plant $P(s)$ as

$$C_s(s) = \tilde{C}_s(s + \xi) = \frac{-0.6445 - 0.9985s - 2.8841s^2}{0.7994 - 1.3351s - 1.6605s^2} \qquad (29)$$

where the poles of the corresponding closed-loop system are

$$-26.7533, \quad -0.2324, \quad -0.2000, \quad -0.2000 \pm 1.8616j, \quad -0.2891 \pm 0.5430j$$

which shows that all of the closed-loop poles have real part no larger than $-\xi$. Fig. 2 shows the deterministic parameter space, where solid lines denote the possible ξ-shifted stability boundaries which correspond to the critical frequencies, and "o" denotes $\theta^{(2)}$.

B. H_∞ Performance

This example continues the previous one. In particular, we now deal with H_∞ performance. The plant and the choice of the controller parameters are the same. The parameters ϵ and δ are also the same.

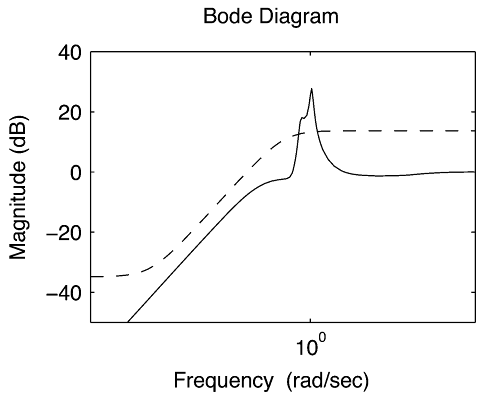

Fig. 3. Sensitivity of the stabilizing controller.

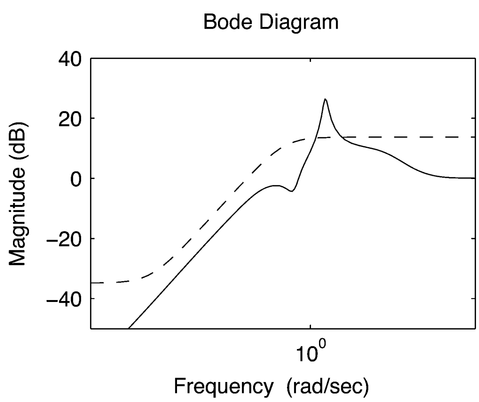

Fig. 4. Sensitivity of the ξ-stabilizing controller.

The sensitivities of the closed loop systems with the controllers (28) and (29) are shown in Figs. 3 and 4 with solid lines. In order to improve it, we introduced a weighting transfer function

$$W(s) = \frac{N_W(s)}{D_W(s)} = \frac{55(1+3s)}{1+800s}$$

where the frequency response of $W^{-1}(s)$ is also shown in Figs. 3 and 4 as dashed line.

We used the same controller parametrization (27). That is, we set $\beta_0 = 1$ and randomly generated $\alpha_0 \in [-5, 5], \beta_2 \in [-5, 5]$, and $\mu_0 \in [-5, 5]$ according to uniform distributions. The parameters in Theorem 1 were chosen as $\epsilon = 0.01$ and $\delta = 0.01$, which are also used in the previous example. The grid points were selected as $\{0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, 3\pi/2, 7\pi/4\}$ with $n_\phi = n_\phi^u = 8$ so that only the nonnegative critical frequencies are considered in the following.

We run Algorithm 1, together with Algorithm 4, for the search of deterministic parameters θ, obtaining a stabilizing controller that meets the H_∞ performance for $i = 391$. The details are as follows.

The randomized parameters were selected as

$$\eta^{(391)} = [\,-0.5407 \quad 1 \quad -1.2592 \quad -0.3645\,]^T.$$

For this $\eta^{(391)}$, we computed the critical frequencies at the given grid points. That is,

$$\begin{aligned}
\Omega(0) &= \{\,0,\ 8.0139,\ 1.3227, 0.5064,\ 0.0371\,\}\\
\Omega(\pi/4) &= \{\,0,\ 3.9085,\ 1.5812, 0.5361,\ 0.0013\,\}\\
\Omega(\pi/2) &= \{\,0,\ 2.7333,\ 1.7806, 0.5713,\ 0.0000\,\}\\
\Omega(3\pi/4) &= \{\,0,\ 3.9963,\ 1.3760, 0.6084,\ 0.1686,\ 0.0357\,\}\\
\Omega(\pi) &= \{\,0,\ 9.4958,\ 1.0641, 0.6250,\ 0.2454,\ 0.0068\,\}\\
\Omega(5\pi/4) &= \{\,0,\ 19.7605,\ 0.9711, 0.5665,\ 0.2876,\ 0.0012\,\}\\
\Omega(3\pi/2) &= \{\,0,\ 22.0546,\ 1.0271, 0.4931,\ 0.2658\,\}\\
\Omega(7\pi/4) &= \{\,0,\ 15.8042,\ 1.1470, 0.4854,\ 0.1714\,\}.
\end{aligned}$$

For $j = 261$, i.e., when we chose 3.9085 at $\phi = \pi/4$ and 0.5665 at $\phi = 5\pi/4$, we had

$$\bar{\Psi}_{261} = \begin{bmatrix} 1 & -15.2766 \\ 1 & -0.3209 \end{bmatrix}, \bar{\nu}_{261} = \begin{bmatrix} 31.2142 \\ 0.1042 \end{bmatrix}$$

and obtained

$$\theta^{(261)} = [\,-0.5634 \quad -2.0801\,]^T$$

by solving the corresponding linear equation. This was found to be a vertex of $\tilde{\Gamma}$.

Then, we computed the desired direction of the perturbation obtaining

$$\Delta\theta = [\,0.6209 \quad -0.1339\,]^T.$$

With the step parameter $\alpha = 0.05$, we finally derived a stabilizing parameter $\theta_s = \theta^{(261)} + \alpha\Delta\theta$

$$\theta_s = [\,-0.5323 \quad -2.0868\,]^T.$$

Thus, the corresponding controller is given by

$$C_s(s) = \frac{-0.532 - 0.5407s - 2.0868s^2}{1 - 0.3645s - 1.2592s^2} \tag{30}$$

where the real parts of the poles of the corresponding closed-loop system are negative.

Figs. 5 and 6 show the deterministic parameter space, where the lines of the candidate performance boundaries which correspond to the critical frequencies at the grid points, and "o" and "x" denote $\theta^{(261)}$ and θ_s respectively. Fig. 7 shows the sensitivity of the closed loop system with the controller (30) as solid line, where the dashed line describes the frequency response of $W^{-1}(s)$. We therefore see that the controller (30) meets the H_∞ performance specification quite well.

X. Conclusion

In this paper, we studied fixed order stabilization of single-input single-output plants. We presented several randomized and deterministic algorithms for solving three different problems: stabilization and H_∞ performance of a fixed plant, and stabilization of interval plants. A detailed complexity analysis is provided together with two application examples.

The key idea behind the proposed algorithm was to employ even-odd structure of the controller and to classify the design

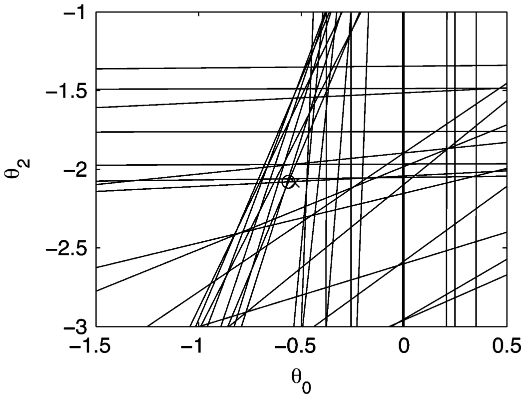

Fig. 5. Deterministic parameter space of the H_∞ controller.

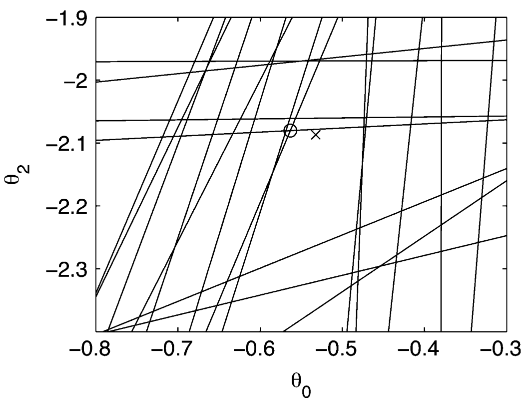

Fig. 6. Deterministic parameter space of the H_∞ controller (magnified).

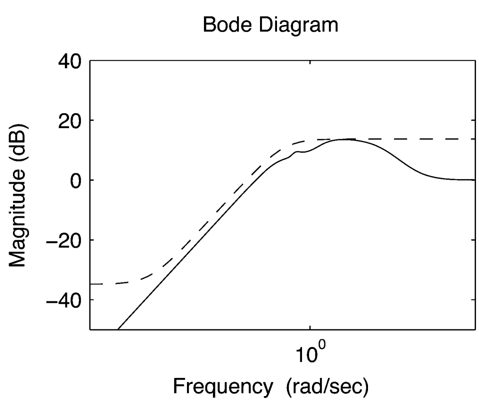

Fig. 7. Sensitivity of the H_∞ controller.

variables into randomized and deterministic parameters. Then, we used randomized algorithms for the former parameters and developed efficient algorithms for the latter. This idea is considered to be effective in various control problems. For the special case of interval polynomials a mixed approach is developed in [32] to solve the so-called "one-in-a-box problem." The same idea may be applicable also to BMI formulations [33] if we split the decision variables into two groups each of which forms an LMI. That is, we can proceed with a similar algorithm considering one group as randomized parameters. On the other hand, the controller design in this paper works directly in the space of controller coefficients, and avoids the drawback of parameter augmentation which is required by the BMI formulations dealing with decision variables which are artificially constructed.

Finally, we would like to mention a recent approach based on nonsmooth optimization [34], which is applicable to many problems such as fixed order controller design and BMI optimization. Notice, however, that this approach is basically a local search gradient-based method which has no guarantee to find a solution even if the problem itself is solvable. On the other hand, the mixed approach proposed in this paper, and in Section III in particular, has such guarantee, of course only in a probabilistic sense.

Subsequent research will be carried on along several directions. In particular, we plan to extend the results of this paper to stabilization of plants affected by a delay [35] and to classes of multi-input multi-output systems.

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